Nearly Ordinary Galois Deformations over Arbitrary Number Fields

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Abstract

Let K be an arbitrary number field, and let $\rho : \mathrm{Gal}(\overline{K}/K) \to GL_2(E)$ be a nearly ordinary irreducible geometric Galois representation. In this paper, we study the nearly ordinary deformations of ρ. When K is totally real and ρ is modular, results of Hida imply that the nearly ordinary deformation space associated to ρ contains a Zariski dense set of points corresponding to "automorphic" Galois representations. We conjecture that if K is *not* totally real, then this is never the case, except in three exceptional cases, corresponding to: (1) "base change", (2) "CM" forms, and (3) "even" representations. The latter case conjecturally can only occur if the image of ρ is finite. Our results come in two flavours. First, we prove a general result for Artin representations, conditional on a strengthening of Leopoldt's conjecture. Second, when K is an imaginary quadratic field, we prove an unconditional result that implies the existence of "many" positive dimensional components (of certain deformation spaces) that do not contain infinitely many classical points. Also included are some speculative remarks about "p-adic functoriality", as well as some remarks on how our methods should apply to n-dimensional representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ when $n > 2$.

1 Introduction

Let K be a number field, $\overline{K}/K$ be an algebraic closure of K, and p a prime that splits completely in K. Fix

$$\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E),$$

a continuous "nearly ordinary" (see Definition 2.2 below) absolutely irreducible p-adic Galois representation unramified outside a finite set of places of K, where E is a field extension of $\mathbf{Q}_p$.

Choose a lattice for the representation ρ, and let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathcal{O}_E/\mathfrak{m}_E)$ denote the associated residual representation. Fixing S a finite set of places of K that include the primes dividing p, the primes of ramification for ρ, and all archimedean places, we may view $\overline{\rho}$ as homomorphism

$$\overline{\rho} : G_{K,S} \to \mathrm{GL}_2(\mathcal{O}_E/\mathfrak{m}_E)$$

where $G_{K,S}$ is the quotient group of $\mathrm{Gal}(\overline{K}/K)$ obtained by division by the closed normal subgroup generated by the inertia groups at all primes of K not in S.

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Let us suppose, for now, that $\overline{\rho}$ is absolutely irreducible. Then we may form R, the universal deformation ring of continuous nearly ordinary deformations of the Galois representation $\overline{\rho}$ (unramified outside S); this is a complete noetherian local ring that comes along with a *universal* nearly ordinary deformation of our representation $\overline{\rho}$,

$$\rho^{\mathrm{univ}} : G_{K,S} \to \mathrm{GL}_2(R),$$

from which we may recover ρ as a specialization at some E-valued point of $\mathrm{Spec}(R)$. For reference, see section 30 of [27].

Suppose that K is totally real, ρ is attached to a Hilbert modular form π of regular weight, p splits completely in K and $\rho|_{D_v}$ is nearly ordinary for all $v|p$. In this situation, a construction of Hida [17, 18] gives us a ("Hida-Hecke"-algebra) quotient $R \to \mathbf{T}$ that is a finite flat algebra over the (n-parameter) affine coordinate ring of weights. Consequently, we have a corresponding "n-parameter" family of nearly ordinary Galois representations which project onto a component of (p-adic) weight space, and which also contain a Zariski dense set of classical (automorphic) representations.

The situation is quite different in the case where K is not totally real. Although our interest in this paper is mainly limited to the study of Galois representations rather than automorphic forms, to cite one example before giving the main the theorem of this article, we have:

Theorem 1.1. *Let $K = \mathbf{Q}(\sqrt{-2})$, and let $\mathfrak{N} = 3 - 2\sqrt{-2}$. Let $\mathbf{T}$ denote the nearly ordinary 3-adic Hida algebra of tame level $\mathfrak{N}$. Then the affine scheme $\mathrm{Spec}(\mathbf{T})$ only contains finitely many classical points and moreover (is nonempty and) has pure relative dimension one over $\mathrm{Spec}(\mathbf{Z}_p)$.*

This is proved in subsection 8.5 below. Although our general aim is to understand deformation spaces of nearly ordinary Galois representations, our primary focus in this article is the close study of *first order* deformations of a given irreducible representation ρ; that is, we will be considering the tangent space of the space of (nearly ordinary) deformations of ρ, $\mathrm{Tan}_\rho(R)$. For most of this article we will focus on the case where ρ factors through a finite quotient of $\mathrm{Gal}(\overline{K}/K)$ — we call such representations *Artin representations* — and where the prime p splits completely in K. If $\mathfrak{m}$ denotes the prime ideal of R corresponding to the representation ρ, we will be looking at the linear mapping of E-vector spaces

$$\tau : \mathrm{Tan}_{\mathfrak{m}}(R) \longrightarrow \mathrm{Tan}_0(W),$$

the canonical projection to the tangent space (at weight 0) of the space of p-adic weights. The projective space $P\mathrm{Tan}_0(W)$ of lines in this latter E-vector space, $\mathrm{Tan}_0(W)$, has a natural descent to $\mathbf{Q}$. A (nonzero) infinitesimal weight $w \in P\mathrm{Tan}_0(W)$ that generates a line that is $\mathbf{Q}$-rational with respect to this underlying $\mathbf{Q}$-structure will be called a **classical infinitesimal weight**. Similarly, a (first-order) deformation of ρ possessing a classical infinitesimal weight will be called an **infinitesimally classical deformation**. (We restrict our attention to deformations of fixed determinant to avoid "trivial" instances of infinitesimally classical deformations arising from twisting by characters.) We feel that (in the case where K is not totally real) the mere occurrence of an infinitesimally classical deformation implies very strong consequences about the initial representation ρ. Our main result is that this is indeed so if we assume a hypothesis that we call the *Strong Leopoldt Conjecture*. Specifically, we prove the following:

Theorem 1.2. *Assume the Strong Leopoldt Conjecture. Let p be prime, and let $K/\mathbf{Q}$ be a Galois extension in which p splits completely. Suppose that $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E)$ is continuous,* *irreducible, nearly ordinary, has finite image, and admits infinitesimally classical deformations. Then either:*

1. *There exists a character χ such that $\chi \otimes \rho$ descends either to an odd representation over a totally real field, or descends to a field containing at least one real place at which $\chi \otimes \rho$ is even.*
2. *The projective image of ρ is dihedral, and the determinant character descends to a totally real field $H^+ \subseteq K$ with corresponding fixed field H such that*
   - (i) *H/H^+ is a CM extension.*
   - (ii) *At least one prime above p in H^+ splits in H.*

Remarks:

1. The restriction that $K/\mathbf{Q}$ be Galois is very mild since one is free to make a base change as long as ρ remains irreducible.
2. The *strong Leopoldt conjecture* (§3) is, as we hope to convince our readers, a natural generalization of the usual Leopoldt conjecture, and is related to other p-adic transcendence conjectures such as the p-adic form of Schanuel's conjecture.
3. We expect that the same conclusion holds more generally for p-adic representations $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E)$ that do not have finite image, but otherwise satisfy the conditions of Theorem 1.2. Explicitly,

Conjecture 1.3. *Let p be prime, and let $K/\mathbf{Q}$ be a Galois extension in which p splits completely. Suppose that $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E)$ is continuous, irreducible, nearly ordinary, is unramified except at finitely many places, and admits infinitesimally classical deformations. Then either:*

1. *There exists a character χ such that $\chi \otimes \rho$ descends either to an odd representation over a totally real field, or descends to a field containing at least one real place at which $\chi \otimes \rho$ is even.*
2. *The projective image of ρ is dihedral, and the determinant character descends to a totally real field $H^+ \subseteq K$ with corresponding fixed field H such that*
   - (i) *H/H^+ is a CM extension.*
   - (ii) *At least one prime above p in H^+ splits in H.*

For some evidence for this, see Theorem 1.5 below. We refer to the first case as "base change" and the second case as "CM."

In the special case where K is a quadratic imaginary field, the E-vector space of infinitesimal weights, $\mathrm{Tan}_0(W)$, is then two-dimensional, and the nontrivial involution (i.e., complex conjugation) of $\mathrm{Gal}(K/\mathbf{Q})$ acting on $\mathrm{Tan}_0(W)$ has unique eigenvectors (up to scalar multiplication) with eigenvalues $+1$ and -1; refer to the points they give rise to in $P\mathrm{Tan}_0 W$ as the *diagonal* infinitesimal weight and the *anti-diagonal*, respectively.

Corollary 1.4. *Assume the Strong Leopoldt Conjecture. Let K be a quadratic imaginary field and p a prime that splits in K. Let $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E)$ satisfy the hypotheses of 1.2, and in particular assume that ρ admits an infinitesimally classical deformation. Assume that the image of ρ is finite and non-dihedral. Then one of the two cases below holds:*

1. *The representation ρ admits an infinitesimally classical deformation with diagonal infinitesimal weight and there exists a character χ such that $\chi \otimes \rho$ descends to an odd representation over $\mathbf{Q}$, or*

2. *the representation ρ admits an infinitesimally classical deformation with anti-diagonal infinitesimal weight and there exists a character χ such that $\chi \otimes \rho$ descends to an even representation over $\mathbf{Q}$.*

The above Corollary 1.4 follows from Theorem 1.2 together with the discussion of section 8.2.

Our result (Theorem 1.2), although comprehensive, is contingent upon a conjecture, and is only formulated for representations with finite image. To offer another angle on the study of deformation spaces (of nearly ordinary Galois representations) containing few classical automorphic forms, we start with a *residual* representation $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{GL}_2(\mathbf{F})$ where $\mathbf{F}$ is a finite field of characteristic p, and $\overline{\rho}$ is irreducible and nearly ordinary at $v|p$ (and satisfies some supplementary technical hypotheses). In section 7, we prove the following complement to Theorem 1.2 when K is an imaginary quadratic field:

Theorem 1.5. *Let K be an imaginary quadratic field in which p splits. Let $\mathbf{F}$ be a finite field of characteristic p. Let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{GL}_2(\mathbf{F})$ be a continuous irreducible Galois representation, let $\chi : \mathrm{Gal}(\overline{K}/K) \rightarrow \mathbf{F}^{\times}$ be the mod-p cyclotomic character, and assume the following:*

1. *The image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbf{F})$, and $p \geq 5$.*
2. *If $v|p$, then $\overline{\rho}$ is nearly ordinary at v and takes the shape: $\rho|_{I_v} = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}$, where $\psi \neq 1, \chi^{-1}$, and $*$ is très ramifiée if $\psi = \chi$.*
3. *If $v \nmid p$, and $\overline{\rho}$ is ramified at v, then $H^2(G_v, \mathrm{ad}^0(\overline{\rho})) = 0$.*

Then there exists a Galois representation: $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/K) \rightarrow \mathrm{GL}_2(W(\mathbf{F})[[T]])$ lifting $\overline{\rho}$ such that:

1. *The image of ρ contains $\mathrm{SL}_2(W(\mathbf{F})[[T]])$.*
2. *ρ is unramified outside some finite set of primes Σ. If $v \in \Sigma$, and $v \nmid p$, then $\rho|D_v$ is potentially semistable; if $v|p$, then $\rho|D_v$ is nearly ordinary.*
3. *Only finitely many specializations of ρ have parallel weight.*

The relation between Theorem 1.5 and Conjecture 1.3 is the following. Standard facts ([12], §III, p.59–75) imply that the cuspidal cohomology of $\mathrm{GL}(2)_{/K}$ vanishes for non-parallel weights. Thus, if a one parameter family of nearly ordinary deformations of $\overline{\rho}$ arise from classical cuspidal automorphic forms over K, they *a priori* must have all have parallel weight. Since families of parallel weight have rational infinitesimal Hodge-Tate weights, Conjecture 1.3 implies that such families are CM or arise from base change (and hence only exist if $\overline{\rho}$ is also of this form). In contrast, Theorem 1.5 at least guarantees the *existence* of non-parallel families for *any* $\overline{\rho}$ (including those arising from base change).

Although we mostly restrict our attention to the study of Galois representations, the general philosophy of $R = \mathbf{T}$ theorems predicts that the ordinary Galois deformation rings considered in this paper are isomorphic to the ordinary Hecke algebras constructed by Hida in [16]. Thus we are inclined to believe the corresponding automorphic conjectural analogs of Conjecture 1.3 and Theorem 1.5.

We now give a brief outline of this paper. In section 2, we recall some basics from the deformation theory of Galois representations, and define the notion of infinitesimal Hodge-Tate weights. In section 3 we present our generalization of the Leopoldt conjecture. In section 4 we prove some lemmas regarding group representations, and in sections 5 and 6, we prove Theorem 1.2. We prove Theorem 1.5 in section 7. In section 8, we speculate further on the connection between nearly ordinary Galois representations, Hida's ordinary Hecke algebra, and classical automorphic forms. We also describe some other problems which can be approached by our methods, for example: establishing the rigidity of three dimensional Artin representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. In section 8 we also give a proof of Theorem 1.1, and in the Appendix (section A) we prove some structural results about generic modules used in sections 3 and 5.

1.1 Speculations on p-adic functoriality

It has long been suspected that torsion classes arising from cohomology of arithmetic quotients of symmetric spaces give rise to Galois representations, even when the associated arithmetic quotients are not Shimura varieties. In the context of $\mathrm{GL}(2)_{/K}$ for an imaginary quadratic field K, some isolated examples for $K = \mathbf{Q}(\sqrt{-1})$ were first noted in [8]. The first conjectures for arbitrary reductive groups G (and coefficient systems over a finite field) were formulated by Ash [1], at least for $G = \mathrm{GL}(n)_{/\mathbf{Q}}$. Let $G_{/\mathbf{Q}}$ denote an arbitrary reductive group that is split at p, K_∞ a maximal compact of $G(\mathbf{R})$, and T_G a maximal torus of G. After fixing a tame level, a construction of Hida assembles (ordinary) torsion classes (relative to some choice of local system) for arithmetic quotients $G(\mathbf{Q})\backslash G(\mathbf{A})/K^\circ_\infty K$, as K varies over p-power levels relative to the fixed tame level. If $\mathbf{T}_G$ denotes the corresponding algebra of endomorphisms generated by Hecke operators on this space (not to be confused with the torus, T_G), then $\mathbf{T}_G$ is finitely generated over the space of weights $\Lambda_G = \mathbf{Z}_p[[T_G(\mathbf{Z}_p)]]$. Hida's control theorems [19][1] imply that for (sufficiently) regular weights κ in $\mathrm{Hom}(\Lambda, \mathbf{C}_p)$, the specialization of $\mathbf{T}_G$ to κ recovers the classical space of ordinary automorphic forms of that weight. It should be noted, however, that Corollary 1.4 implies that $\mathbf{T}_G$ can be large even if the specialization to every regular weight of Λ_G is zero, and that $\mathbf{T}_G$ should be thought of as an object arising from Betti cohomology (taking torsion cohomology classes into consideration as well) rather than anything with a more standard automorphic interpretation, such as $(\mathfrak{g}, K)$ cohomology. Suitably adapted and generalized, the conjectures alluded to above imply the existence of a map $R_G \to \mathbf{T}_G$, where R_G denotes an appropriate (naturally defined) Galois deformation ring. One of our guiding principles is that the map $R_G \to \mathbf{T}_G$—for suitable definitions of R_G and $\mathbf{T}_G$—should *exist* and be an isomorphism. If $G = \mathrm{GL}(2)_{/\mathbf{Q}}$, then this conjecture has (almost) been completely established, and many weaker results are also known for $G = \mathrm{GL}(2)_{/K}$ if K is totally real (such results are usually conditional, for example, on the modularity of an associated residual representation $\overline{\rho}$).

At first blush, our main results appear to be negative, namely, they suggest that the ring $\mathbf{T}_G$ — constructed from Betti cohomology — has very little to do with automorphic forms. We would like, however, to suggest a second possibility. Let us define a (nearly ordinary) emph$p$-adic automorphic form for G to be a characteristic zero point of $\mathrm{Spec}(\mathbf{T}_G)$. Then we conjecture that p-adic automorphic forms are subject to the same principle of functoriality as espoused by Langlands for classical automorphic forms. Such a statement is (at the moment) imprecise and vague — even to begin to formalize this principle would probably require a more developed theory of p-adic local Langlands than currently exists. And yet, in the case of $\mathrm{GL}(2)_{/K}$ for an imaginary quadratic

[1]Which presumably hold in full generality, and have been established (by Hida himself) in many cases, $G = \mathrm{GL}(2)_{/K}$, $\mathrm{GSp}(4)_{/\mathbf{Q}}$ — the examples that arise in the discussion — included.

field K, we may infer two consequences of these conjectures which are quite concrete, and whose resolutions will probably hold significant clues as to what methods will apply more generally.

The first example concerns the possibility of p-adic automorphic lifts. Fix an imaginary quadratic field K. Let $G = \mathrm{GL}(2)_{/K}$ and $H = \mathrm{GSp}(4)_{/\mathbf{Q}}$. Given a p-adic automorphic form π for G arising from a characteristic zero point of $\mathrm{Spec}(\mathbf{T}_G)$, a p-adic principle of functoriality would predict the following: if π satisfies a suitable condition on the central character, it lifts to a p-adic automorphic form for H. The existence of such a lift is most easily predicted from the Galois representation ρ (conjecturally) associated to π. Namely, if $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E)$ is a representation with determinant χ, and $\chi = \phi\psi^2$ for characters ϕ, ψ of K, where ϕ lifts to a character of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, then $\mathrm{Ind}^{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}_{\mathrm{Gal}(\overline{K}/K)}(\rho \otimes \psi^{-1})$ has image landing inside $\mathrm{GSp}_4(E)$ for some suitable choice of symplectic form, and will be the Galois representation associated to the conjectural p-adic lift of π to H. If the original p-adic automorphic form π is actually *classical*, then this lifting is known to exist by work of Harris, Soudry, and Taylor [13]. The group $H = \mathrm{GSp}(4)_{/\mathbf{Q}}$ has the convenient property that the corresponding arithmetic quotients are PEL Shimura varieties. Consequently, there is a natural source of Galois representations, and, more relevantly for our purposes, the Hida families associated to cuspidal ordinary forms of middle degree (3, in this case) cohomology form a Hida family $\mathbf{T}_H$ which is *flat* over Λ_H. Yoneda's lemma implies the existence of maps between various rings: from Galois induction, we obtain a surjective map $R_H \to R_G$, and from p-adic functoriality, we predict the existence of a surjective map $\mathbf{T}_H \to \mathbf{T}_G$. All together we have a diagram:

$$\begin{tikzcd} R_H \arrow[r] \arrow[d] & \mathbf{T}_H \arrow[d, dotted] \\ R_G \arrow[r, dotted] & \mathbf{T}_G, \end{tikzcd}$$

where only the solid arrows are known to exist. Naturally, we conjecture that the horizontal arrows are isomorphisms, and that the diagram is commutative. A calculation on tangent spaces suggests that the relative dimensions of R_H and R_G over $\mathbf{Z}_p$ are 2 and 1 respectively (factoring out twists), and thus $\mathrm{Spec}(\mathbf{T}_G)$ should "sit" as a *divisor* inside $\mathrm{Spec}(\mathbf{T}_H)$.

If this picture is correct, we would be facing the following curious situation, in light of Corollary 1.4: the space $\mathrm{Spec}(\mathbf{T}_H)$ has a dense set of classical automorphic points, and yet the divisor $\mathrm{Spec}(\mathbf{T}_G)$ contains components with only finitely many classical points. Thus, although the points of $\mathrm{Spec}(\mathbf{T}_G)$ are not approximated by classical automorphic forms for G (they cannot be, given the paucity of such forms), they *are*, in some sense, approximated by classical (indeed, holomorphic) automorphic forms for H. Such a description of $\mathrm{Spec}(\mathbf{T}_G)$ would allow one to study classical points of $\mathrm{Spec}(\mathbf{T}_G)$ (for example, automorphic forms π associated to modular elliptic curves over K) by variational techniques.

Given the above speculation, there may be several approaches to defining p-adic L-functions (of different sorts) on $\mathrm{Spec}(\mathbf{T}_G)$. The first approach might follow the lead of the classical construction using modular symbols, i.e., one-dimensional homology of the appropriate symmetric spaces, but would make use of all the Betti homology including torsion classes. The second approach might lead us to to a p-adic analogue of the "classical" (degree 4) Asai L-function by constructing p-adic L-functions on $\mathrm{Spec}(\mathbf{T}_H)$ (banking on the type of *functoriality* conjectured above, and using the density of classical points on $\mathrm{Spec}(\mathbf{T}_H)$) and then restricting.

One may also ask for a characterization of the (conjectured) divisor given by the functorial lifting of $\mathrm{Spec}(\mathbf{T}_G)$ to $\mathrm{Spec}(\mathbf{T}_H)$. For a *classical* automorphic form π on H, the Tannakian formalism of L-groups (or, perhaps equally relevantly, a theorem of Kudla, Rallis, and Soudry [21]) implies that π arises from G if and only if the degree 5 (standard) L-function $L(\pi \otimes \chi, s)$ has a pole at $s = 1$. Hence, speculatively, we conjecture that the divisor $\mathrm{Spec}(\mathbf{T}_G)$ in $\mathrm{Spec}(\mathbf{T}_H)$ is cut out by the polar divisor of a p-adic (two variable, and as yet only conjectural) degree-5 L-function on $\mathrm{Spec}(\mathbf{T}_H)$ at $s = 1$. Moreover, the residue at this pole will be the (conjectural) special value of the Asai L-function interpolated along $\mathrm{Spec}(\mathbf{T}_G)$.

A second prediction following from p-adic functoriality is as follows. Once more, let $G = \mathrm{GL}(2)_{/K}$, where K is an imaginary quadratic field, and now let G' denote an inner form of G, specifically, arising from a quaternion algebra over K. Then results of Hida [15] provide us with *two* ordinary Hecke algebras $\mathbf{T}_G$ and $\mathbf{T}_{G'}$ which are one dimensional over $\mathbf{Z}_p$. The theorem of Jacquet-Langlands implies that any classical point of $\mathrm{Spec}(\mathbf{T}_{G'})$ corresponds (in the usual sense) to a classical point of $\mathrm{Spec}(\mathbf{T}_G)$. If classical points were dense in $\mathrm{Spec}(\mathbf{T}_{G'})$, then one would expect that this correspondence would force $\mathrm{Spec}(\mathbf{T}_{G'})$ to be identified with a subvariety of $\mathrm{Spec}(\mathbf{T}_G)$. In certain situations this expectation can be translated into a formal argument, and one obtains a "p-adic Jacquet-Langlands" for p-adic automorphic forms as a consequence of the classical theorem. This is because, in situations arising from Shimura varieties, eigencurves are uniquely determined by their classical points, and are thus rigid enough to be "independent of their constructor" [3], §7. However, for our G and G', Corollary 1.4 exactly predicts that classical points will *not* be dense, and hence no obvious relation between $\mathrm{Spec}(\mathbf{T}_G)$ and $\mathrm{Spec}(\mathbf{T}_{G'})$ follows from classical methods. Instead, we predict the existence of a genuine *p-adic* Jacquet-Langlands for GL(2) which does *not* follow from the usual classical theorem, nor (in any obvious way) from any general classical form of functoriality, and that, by keeping track of the action of Hecke, this correspondence will identify $\mathrm{Spec}(\mathbf{T}_{G'})$ as a subvariety of $\mathrm{Spec}(\mathbf{T}_G)$.

It should be noted that the phenomena of non-classical points can even be seen even in $\mathrm{GL}(1)_{/K}$, when K is neither totally real nor CM. Namely, there exist Galois representations $\mathrm{Gal}(\overline{K}/K) \to E^*$ that cannot be approximated by Galois representations arising from algebraic Hecke characters. For example, suppose the signature of K is (r_1, r_2) with $r_2 > 0$, and that K does not contain a CM field. Then the only Galois representations unramified away from p arising from algebraic Hecke characters are (up to a bounded finite character) some integral power of the cyclotomic character. The closure of such representations in deformation space has dimension one. On the other hand, the dimension of the space of one dimensional representations of $\mathrm{Gal}(\overline{K}/K)$ unramified away from p has dimension *at least* $r_2 + 1$, equality corresponding to the Leopoldt conjecture. (See also section 8.4.) One reason for studying $G = \mathrm{GL}(2)_{/K}$, however, is that when K is a CM field, G is related via functorial maps to groups associated to Shimura varieties, such as $\mathrm{GSp}(4)_{/K^+}$ or $U(2,2)_{/K}$. Even if the general conjectural landscape of arithmetic cohomology beyond Shimura varieties requires fundamentally new ideas, some progress should be possible with the classical tools available today if one is allowed some access to algebraic geometry. The other simplest case to consider is to try to produce liftings from p-adic automorphic forms on $\mathrm{GL}(1)_{/L}$ to $\mathrm{GL}(2)_{/K}$, where K is an imaginary quadratic field, and L/K a quadratic extension. To put oneself into a genuinely p-adic situation, one should insist that L is not a CM field, equivalently, that $L/\mathbf{Q}$ is not biquadratic.

Due to the lack—-at present—-of substantial progress on these conjectures, we shall limit discussion of these matters, in this paper at least, to this section. However, we believe that it is useful to view the results of this paper in the light of these speculations, as they are a first step to acknowledging that a "p-adic" approach will be necessary to studying automorphic forms over $\mathrm{GL}(2)_{/K}$.

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Contents

2 Deformation Theory

2.1 Nearly Ordinary Selmer Modules

Fix a prime number p, and let $(A, \mathfrak{m}_A, k)$ be a complete local noetherian (separated) topological ring. We further suppose that the residue field k is either

- an algebraic extension of $\mathbf{F}_p$, in which case A is endowed with its standard $\mathfrak{m}$-adic topology, or
- a field of characteristic zero complete with respect to a p-adic valuation (such a field we will simply call a *p-adic field*, and also denote by E) in which case we require that the maximal ideal be closed and that the reduction homomorphism $A \to k$ identifies the topological field A/m with k.

If M is a free A-module of rank d, $\mathrm{Aut}_A(M) \simeq \mathrm{GL}_d(A)$ is canonically endowed with the structure of topological group. If ℓ is a prime number different from p, neither of the topological groups M or $\mathrm{Aut}_A(M)$ contain infinite closed pro-ℓ subgroups.

Definition 2.1. *Let V_A be a free rank two A-module. By a* **line** *$L_A \subset V_A$ one means a free rank one A-submodule for which there is a complementary free rank one A-submodule $\tilde{L}_A \subset V_A$, so that $V_A = L_A \oplus \tilde{L}_A$.*

Let $K/\mathbf{Q}$ be an algebraic number field, and assume that p splits completely in K.

Definition 2.2. *Let V_A be a free rank two A-module with a continuous A-linear action of $G_K := \mathrm{Gal}(\overline{K}/K)$, and let*

$$\rho_A : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}_A(V_A) \simeq \mathrm{GL}_2(A)$$

be the associated Galois representation. Such a representation ρ is said to be nearly ordinary *at a prime $v|p$ if there is a choice of decomposition group at v, $\mathrm{Gal}(\overline{K}_v/K_v) \simeq D_v \hookrightarrow G_K$, and a line $L_{A,v} \subset V_A$ stabilized by the action (under ρ_A) of D_v. If D_v stabilizes more than one line, we will suppose that we have chosen one such line $L_{A,v}$, which we refer to as the* **special line** *at v.*

Remark: The line $L_{A,v}$ will depend on the choice of $D_v \subset G_K$ but if one choice of decomposition group D_v at v stabilizes a line $L_{A,v}$ then any other choice D'_v is conjugate to D_v, i.e., $D'_v = gD_vg^{-1}$ for some $g \in G_K$ and therefore will stabilize the transported line $L'_{A,v} := gL_{A,v}$. Let $I_v \subset D_v$ denote the inertia subgroup of the chosen decomposition group $D_v \subset G_K$ at the prime v.

If ρ_A is nearly ordinary at v, then there we have the decomposition

$$0 \to L_{A,v} \to V_A \to V/L_{A,v} \to 0$$

of $A[D_v]$-modules. Suppose we express a G_K-representation ρ_A that is nearly ordinary at v as a matrix representation (with respect to a basis of V_A where the first basis element lies in the special line). Then, the representation ρ restricted to D_v takes the shape:

$$\rho_A|_{D_v} = \begin{pmatrix} \psi_{A,v} & * \\ 0 & \psi'_{A,v} \end{pmatrix}$$

where $\psi_{A,v} : D_v \to A^*$ is the character of D_v given by its action on $L_{A,v}$, and $\psi'_{A,v}$ the character of D_v given by its action on $V_A/L_{A,v}$.

Let us denote the reductions of the characters $\psi_{A,v}, \psi'_{A,v}$ mod $\mathfrak{m}_A$ by ψ_v, ψ'_v.

Definition 2.3. *We say that the representation ρ_A is* **distinguished** *at v if ρ_A is nearly ordinary at v, and the pair of residual characters $\psi_v, \psi'_v : D_v \to k^*$ are* distinct. *If ρ is distinguished at v there are at most two D_v-stable lines to choose as our special "$L_{A,v}$", and if, in addition, V_A is indecomposable as D_v-representation, there is only one such choice.*

Remark: Let $A \to A'$ be a continuous homomorphism (and not necessarily local homomorphism) of the type of rings we are considering. If V_A is a free rank two A-module endowed with continuous G_K-action, and if we set $V_{A'} := V_A \otimes_A A'$ with induced G_K-action, then if V_A is nearly ordinary (resp. distinguished) at v so is $V_{A'}$. If V_A is nearly ordinary at all places of K dividing p. with $L_{A,v}$ its special line at v, and we take $L_{A',v} := L_{A,v} \otimes_A A' \subset V_{A'}$ to be the special line at v of $V_{A'}$ for all $v \mid p$, we say that the natural homomorphism $V_A \to V_{A'}$ is a **morphism of nearly ordinary G_K-representations**. In the special case where A' is the residue field k of A and $A \to k$ is the natural projection, suppose we are given V a nearly ordinary, continuous, k-representation of G_K. Then the isomorphism class of the pair $(V_A, V_k \xrightarrow{\nu} V)$ where V_A is a nearly ordinary $A[G_K]$-representation and $\nu : V_k \simeq V$ is an isomorphism of nearly ordinary $k[G_K]$ representations, is called a **deformation of the nearly ordinary $k[G_K]$-representation V to A**.

We now consider the adjoint representation associated to ρ_A; that is, the action of G_K on $\mathrm{Hom}_A(V_A, V_A)$ is given by $g(h)(x) = g(h(g^{-1}x))$ for $g \in G_K$ and $h \in \mathrm{Hom}_A(V_A, V_A)$.

If $W_A := \mathrm{Hom}'_A(V_A, V_A) \subset \mathrm{Hom}_A(V_A, V_A)$ denotes the hyperplane in $\mathrm{Hom}_A(V_A, V_A)$ consisting of endomorphisms of V_A of trace zero (equivalently, endomorphisms whose matrix expression in terms of any basis has trace zero), then W_A is a G_K-stable free A-module of rank three, for which the submodule of scalar endomorphisms $A \subset \mathrm{Hom}_A(V_A, V_A)$ is a G_K-stable complementary subspace.

If v is a finite place of K that is (nearly ordinary, and) distinguished for ρ_A, consider the following D_v-stable flag of free A-submodules of W_A (each possessing a free A-module complement in W_A):

$$0 \subset W^{00}_{A,v} \subset W^0_{A,v} \subset W_A,$$

where

- $W^0_{A,v} := \mathrm{Ker}(\mathrm{Hom}'_A(V_A, V_A) \longrightarrow \mathrm{Hom}_A(L_{A,v}, V_A/L_{A,v}))$, or, equivalently, $W^0_{A,v} \subset W_A$ is the A-submodule of endomorphisms of trace zero that bring the line $L_{A,v}$ into itself. We have the exact sequence of A-modules

$$(*) \qquad 0 \to W^0_{A,v} \to W_A \to W_A/W^0_{A,v} \to 0$$

  and note that since we are in a *distinguished* situation, $W_A/W^0_{A,v} \simeq \mathrm{Hom}_A(L_{A,v}, V_A/L_{A,v})$ is a D_v-representation over A of dimension one given by the character $\psi'_{A,v}\psi^{-1}_{A,v} : D_v \to A^*$ whose associated residual character is nontrivial.
- $W^{00}_{A,v} := \mathrm{Ker}(W^0_{A,v} \longrightarrow \mathrm{Hom}_A(L_{A,v}, V_A))$, or equivalently, $W^{00}_{A,v} := \mathrm{Hom}_A(V_A/L_{A,v}, L_{A,v}) \subset W_A$. The subspace $W^{00}_{A,v} \subset W_A$ can also be described as the subspace of endomorphisms that send the line $L_{A,v}$ to zero and are of trace zero. In the case when $A = k$ is a field, we may describe $W^{00}_{k,v}$ as the k-vector subspace of $W^0_{k,v}$ consisting of nilpotent endomorphisms.

Sometimes we write $W_A = W_{A,v}$, when we wish to emphasize that we are considering the $A[G]$-module W_A as a $A[D_v]$-module. The natural homomorphism $W^0_{A,v} \xrightarrow{\epsilon} \mathrm{Hom}_A(L_{A,v}, L_{A,v}) = \mathrm{End}_A(L_{A,v}) = A$ fits in an exact sequence of D_v-modules

$$0 \to W^{00}_{A,v} \ \to \ W^0_{A,v} \ \xrightarrow{\epsilon} \ A \to 0.$$

Choosing a basis of V_A where the first basis element lies in $L_{A,v}$, we may (in the usual way) identify $\mathrm{Hom}_A(V_A, V_A)$ with the space of 2×2 matrices with entries in A. This identifies $W^0_{A,v}$ with the space of upper triangular matrices of trace zero, and the subspace $W^{00}_{A,v}$ with matrices of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$.

In what follows below we shall be dealing with group cohomology with coefficients in modules of finite type over our topological $\mathbf{Z}_p$-algebras A; these can be computed using continuous co-chains.

Fix $\rho_A : G_K \to \mathrm{Aut}_A(V_A)$ a continuous G_K-representation that is *nearly ordinary* and *distinguished* at all primes $v|p$.

Let Σ denote a (possibly empty) set of primes in K, not containing any prime above p. For each place v in K, we define a local Selmer condition $\mathcal{L}_{A,v} \subseteq H^1(D_v, W_{A,v})$ as follows:

1. If $v \notin \Sigma$ and $v \nmid p$, then $\mathcal{L}_{A,v}$ = the image of $H^1(D_v/I_v, W^{I_v}_{A,v})$ in $H^1(D_v, W_{A,v})$ under the natural inflation homomorphism.
2. If $v \in \Sigma$, then $\mathcal{L}_{A,v} = H^1(D_v, W_{A,v})$.
3. If $v|p$, then let $\mathcal{L}_{A,v} = \mathrm{Ker}(H^1(D_v, W_{A,v}) \to H^1(I_v, W_{A,v}/W^0_{A,v}))$.

Lemma 2.4. *Let v be a prime of K. The following relations hold amongst local cohomology groups.*

1. $H^1\big(D_v/I_v, (W_{A,v}/W^0_{A,v})^{I_v}\big) = 0$.
2. *The natural mapping $H^1(D_v, W^0_{A,v}) \to H^1(D_v, W_{A,v})$ is an injection.*
3. *If $v|p$, the image $H^1(D_v, W^0_{A,v}) \hookrightarrow H^1(D_v, W_{A,v})$ is equal to the submodule*

$$\mathcal{L}_{A,v} = \mathrm{Ker}(H^1(D_v, W_{A,v}) \to H^1(I_v, W_{A,v}/W^0_{A,v})) \subset H^1(D_v, W_{A,v}).$$

Proof. We proceed as follows.

1. Either $(W_{A,v}/W^0_{A,v})^{I_v} = 0$ in which case (1) follows, or else (since v is distinguished) D_v/I_v acts on $(W_{A,v}/W^0_{A,v})^{I_v}$ via a (residually) nontrivial character. Since $D_v/I_v \simeq \hat{\mathbf{Z}}$ it follows that $H^1(D_v/I_v, W_{A,v}/W^0_{A,v}) = 0$, giving (1).
2. Since ρ_A is distinguished at v, the action of D_v on the line $W_{A,v}/W^0_{A,v}$ is via a character that is residually nontrivial, and so $H^0(D_v, W_{A,v}/W^0_{A,v}) = 0$. Therefore, by the long exact cohomological sequence associated to $(*)$ above, we have the equality
$$H^1(D_v, W^0_{A,v}) = \mathrm{Ker}(H^1(D_v, W_{A,v}) \to H^1(D_v, W_{A,v}/W^0_{A,v})) \subset H^1(D_v, W_{A,v})$$
establishing (2).
3. The Hochschild-Serre Spectral Sequence applied to the pair of groups $I_v \lhd D_v$ as acting on $W_{A,v}/W^0_{A,v}$ gives that
$$H^1\big(D_v/I_v, (W_{A,v}/W^0_{A,v})^{I_v}\big) \to H^1(D_v, W_{A,v}/W^0_{A,v}) \to H^1(I_v, W_{A,v}/W^0_{A,v})$$
is exact; (3) then follows from (1) and (2).

$\square$

If $c \in H^1(K, W_A)$, let $c_v \in H^1(D_v, W_{A,v})$ denote is image under the restriction homomorphism to D_v.

Definition 2.5. *The Σ-Selmer A-module $H^1_\Sigma(K, W_A)$ consists of classes $c \in H^1(K, W_A)$ such that $c_v \in \mathcal{L}_{A,v}$ for each finite prime v. We denote this by $H^1_\emptyset(K, W_A)$ when Σ is the empty set.*

Remarks. From the viewpoint of universal (or versal) deformation spaces, the Selmer A-module $H^1_\Sigma(K, W_A)$ (taken as coherent $\mathcal{O}_{\mathrm{Spec}(A)}$-module) is the normal bundle of the A-valued section determined by ρ_A in the appropriate versal deformation space.

From now on in this article (outside of section 7) we shall be exclusively interested in Selmer modules when $A = E$ is a p-adic field. In this setting we shall omit the subscript A, a change that is recorded by the following definition.

Definition 2.6. *If $A = E$ is a p-adic field, we write W, W^0, W^{00} for W_E, W^0_E, W^{00}_E, and $\mathcal{L}_v$ for $\mathcal{L}_{E,v}$.*

Lemma 2.7. *If $v \nmid p$, and if the action of inertia at v on the E-module W_v factors through a finite group (e.g., if v is unramified), then $\mathcal{L}_v = H^1(D_v, W_v)$.*

Proof. We must show that if $v \nmid p$, $H^1(D_v/I_v, W_v^{I_v}) \to H^1(D_v, W_v)$ is surjective. It suffices, then, to show that $H^1(I_v, W_v) = 0$. Let $N_v \lhd I_v$ denote the kernel of the (continuous) action of I_v on W_v. We have by our hypothesis that I_v/N_v is a finite cyclic group. The Hochschild-Serre Spectral Sequence applied to $N_v \lhd I_v$ acting on W_v yields the exact sequence
$$H^1(I_v/N_v, W_v) \to H^1(I_v, W_v) \to H^1(N_v, W_v).$$
Since I_v/N_v is finite and W_v is a vector space over a field of characteristic zero, the left flanking group vanishes. Since the N_v-action on W_v is trivial, we have that $H^1(N_v, W_v) = \mathrm{Hom}_{\mathrm{cont}}(N_v^{\mathrm{ab}}, W_v)$. Since N_v^{ab} is a finite group times a pro-ℓ-group for $\ell \neq p$ it follows that $\mathrm{Hom}_{\mathrm{cont}}(N_v^{\mathrm{ab}}, W_v)$ vanishes. $\square$

Proposition 2.8. *If for all $v \nmid p$ the inertia group at v acts on V through a finite group, then the Selmer E-vector space $H^1_\Sigma(K,W)$ is independent of Σ. In particular, for any Σ (as defined above), we have a canonical isomorphism of E-vector spaces $H^1_\Sigma(K,W) \simeq H^1_\emptyset(K,W)$.*

Proof. By Lemma 2.7 for such primes $v \nmid p$, i.e., when the inertia group at v acts on V_E through a finite group, it follows that $\mathcal{L}_v = H^1(D_v, W_v)$ whether or not v is in Σ. $\square$

Corollary 2.9. *When the action of G_K on V factors through a finite group, for any Σ we have*

$$H^1_\Sigma(K,W) \simeq H^1_\emptyset(K,W).$$

2.2 Universal deformations and Selmer groups

Let $V = V_E$ in the notation of subsection 2.1 above, and let $\rho = \rho_E : G_K \to \mathrm{Aut}_E(V)$ be a continuous E-linear representation unramified outside a finite set of places of K and nearly ordinary and distinguished at all v dividing p. Let Σ be a (possibly empty) set of primes in K as in subsection 2.1. Set S to be the (finite) set consisting of all places v of K that are either in Σ, or divide p, or are archimedean, or are ramified for ρ. We may view ρ as a (continuous) representation

$$\rho : G_{K,S} \longrightarrow \mathrm{Aut}_E(V) \approx \mathrm{GL}_2(E),$$

where $G_{K,S}$ is the quotient group of G_K obtained by division by the closed normal subgroup generated by the inertia groups of primes of K not in S. Suppose, now, that ρ is absolutely irreducible and let $R(S) = R$ denote the ring that is the universal solution to the problem of deforming ρ to two-dimensional nearly ordinary $G_{K,S}$-representations over complete local noetherian topological algebras A with residue field the p-adic field E. Thus we have

$$\rho^{\mathrm{univ}} : G_{K,S} \longrightarrow \mathrm{GL}_2(R),$$

a universal representation (nearly ordinary, and unramified outside S) with residual representation equal to ρ. For this, see §30 of [27] but to use this reference, the reader should note that the case of finite residue fields rather that p-adic fields such as E were explicitly dealt with there, and also although a variety of local conditions are discussed, including the condition of *ordinariness* at a prime v, the local condition of *nearly ordinariness* is not treated, so we must adapt the proofs given there to extend to p-adic residue fields such as E, and also to note that with little change one may obtain same (pro-)representability conclusion—as obtained there for ordinariness—for nearly ordinariness. All this is routine. If $\mathfrak{m}$ denotes the maximal ideal of R, then $R/\mathfrak{m} = E$, and the canonical map $R \to R/\mathfrak{m} = E$ induces (from ρ^{univ}) the representation ρ.

Remark: Let $\mathcal{O}_E \subset E$ be the ring of integers in the complete valued field E, and suppose that $\rho : G_K \to \mathrm{GL}_2(E)$ takes its values in $\mathrm{GL}_2(\mathcal{O}_E)$. Put $\mathbf{F} := \mathcal{O}_E/\mathfrak{m}_E\mathcal{O}_E$ and suppose that the residual representation $\overline{\rho} : G_{K,S} \to \mathrm{GL}_2(\mathbf{F})$ associated to ρ is *irreducible*, (or, more generally, satisfies $\mathrm{End}_{\mathbf{F}}(\overline{\rho}) = \mathbf{F}$). Suppose, furthermore, that $\overline{\rho}|_{D_v}$ is nearly ordinary and distinguished for each $v|p$. Then $\overline{\rho}$ also admits a universal nearly ordinary deformation ring $R(\overline{\rho})$. Consider the map $R(\overline{\rho}) \to \mathcal{O}_E$ corresponding to the representation ρ, and denote the kernel of this homomorphism by P. We may recover R from $R(\overline{\rho})$ as follows. Let $\widehat{R(\overline{\rho})}$ denote the completion of $R(\overline{\rho})[1/p]$ at P. Let

E' denote the residue field of this completion — the field E' is the smallest p-adic subfield of E such that the image of ρ in $\mathrm{GL}_2(E)$ may be conjugated into $\mathrm{GL}_2(E')$. Then there is an isomorphism $R \simeq \widehat{R(\overline{\rho})} \otimes_{E'} E$ (see [20], Prop 9.5).

For each prime v of K not dividing p, and choice of inertia subgroup $I_v \subset D_v \subset \mathrm{Gal}(\bar{K}/K)$ denote by

$$I_{v,\rho} \subset I_v$$

the kernel of the restriction of ρ to I_v. Thus if v is a prime unramified for ρ and not in S, we have that $I_{v,\rho} = I_v$ is also in the kernel of the universal representation ρ^{univ}.

Definition 2.10. *Say that a deformation of ρ to a complete local ring A with residue field E, $\rho_A : G_{K,S} \longrightarrow \mathrm{GL}_2(A)$, has* **restricted ramification** *at a prime v of K if $I_{v,\rho}$ is in the kernel of ρ_A.*

Let R_Σ denote the maximal quotient ring of $R(S)$ with the property that the representation

$$\rho^\Sigma : G_{K,S} \longrightarrow \mathrm{GL}_2(R_\Sigma)$$

induced from ρ^{univ} via the natural projection has the property that it has restricted ramification at all (ramified) primes $v \not| p$ of K that are not in Σ. Visibly, ρ is such a representation, and thus $R_\Sigma \neq 0$. Let $\mathcal{X}_\Sigma = \mathrm{Spec}(R_\Sigma)$. We will be particularly interested in $\mathrm{Tan}_{\mathfrak{m}}\mathcal{X}_\Sigma^{\mathrm{det}}$, the Zariski tangent space of $\mathcal{X}_\Sigma$ at the maximal ideal $\mathfrak{m}$ (corresponding to ρ).

It is often convenient to also fix the determinant of the representation. Let $\mathcal{X}_\Sigma^{\mathrm{det}} \subset \mathcal{X}_\Sigma$ denote the closed subscheme cut out by the extra requirement that the kernel (in $G_{K,S}$) of the determinant of any A-valued point of $\mathcal{X}_\Sigma^{\mathrm{det}}$ is equal to the kernel of the determinant of ρ; we view this subscheme as *the universal nearly ordinary deformation space of $\rho : G_{K,S} \to \mathrm{GL}_2(E)$ with restricted ramification outside p and Σ, and with fixed determinant.* If $\Sigma = \emptyset$ then we say that we are in a **minimally ramified** situation.

Let $\mathcal{C}$ denote the category of Artinian E-algebras, and consider the functor $\mathcal{D}_{\rho,\Sigma} : \mathcal{C} \to \mathrm{Sets}$ that associates to an Artinian E-algebra A with residue field E the set $\mathcal{D}_{\rho,\Sigma}(A)$ of all deformations of the Galois representation ρ to A that are nearly ordinary and of restricted ramification at primes not dividing p and outside Σ.

If we impose the extra condition that the deformation have fixed determinant, we obtain a sub-functor that we denote $\mathcal{D}_{\rho,\Sigma}^{\mathrm{det}}$:

$$A \longmapsto \mathcal{D}_\Sigma^{\mathrm{det}}(A) \subset \mathcal{D}_\Sigma(A).$$

Proposition 2.11. *The scheme $\mathcal{X}_\Sigma^{\mathrm{det}}$ (pro-)represents the functor $\mathcal{D}_\Sigma^{\mathrm{det}}$: there is a functorial correspondence $\mathcal{X}_\Sigma^{\mathrm{det}}(A) = \mathcal{D}_\Sigma^{\mathrm{det}}(A)$.*

Proof. This follows in a straightforward way from the definitions, given the discussion above (compare §30 of [27], or [10]). ∎

Let $A = E[\epsilon] = E \oplus \epsilon \cdot E$ be the (Artinian local) E-algebra of dual numbers, where $\epsilon^2 = 0$. There is a natural way of supplying the sets $\mathcal{D}_\Sigma^{\mathrm{det}}(A) \subset \mathcal{D}_\Sigma(A)$ with E-vector space structures (consult loc. cit. for this) and we have the following natural isomorphisms.

Proposition 2.12. *We have identifications of E-vector spaces*

$$\mathrm{Tan}_{\mathfrak{m}}\mathcal{X}_{\Sigma}^{\det} \simeq \mathcal{D}_{\Sigma}^{\det}(A) \simeq H^1_{\Sigma}(K, W).$$

Suppose, moreover, that for all $v \not| p$, the action of the inertia group I_v on W factors through a finite quotient group. Then $H^1_{\Sigma}(K, W)$ is independent of Σ.

Proof. The first identification above is standard. See, for example, §30 of [27], or [10]. The second identification follows from a computation, the format of which is standard; it follows from expressing of $\mathrm{Aut}_{E[\epsilon]}(V \oplus \epsilon \cdot V) \approx \mathrm{GL}_2(E[\epsilon])$ as a semi-direct product of $\mathrm{Aut}_E(V)$ by $\mathrm{End}_E(V) = W \oplus E$, and then noting that a lifting ρ_A of the homomorphism ρ to A provides us, in the usual manner, with a continuous 1-cocycle $c(\rho_A)$ on G_K with values in $W \oplus E$, and if the determinant is *constant* the projection of $c(\rho_A)$ to the second factor vanishes. The underlying cohomology class, $h(\rho_A)$, of $c(\rho_A)$ determines and is determined by the deformation class of the homomorphism ρ_A. The assignment $\rho_A \mapsto h(\rho_A)$ induces a natural embedding

$$\mathcal{D}_{\Sigma}^{\det}(A) \subset H^1(K, W) \subset H^1(K, W) \oplus H^1(K, E) = H^1(K, \mathrm{End}_E(V)).$$

If $h \in H^1(K, W)$ and v is a place, denote its restriction to D_v by $h_v \in H^1(D_v, W_v)$.

Sublemma 1. *The subspace $\mathcal{D}_{\Sigma}^{\det}(A) \subset H^1(K, W)$ consists of the set of cohomology classes $h \in H^1(K, W)$ such that for each place v, h_v lies in $\mathcal{L}'_v \subset H^1(D_v, W_v)$ where*

- *if $v \mid p$ then $\mathcal{L}'_v = \mathcal{L}_v$,*
- *if v is in Σ, then $\mathcal{L}'_v = H^1(D_v, W_v)$, and*
- *if v is none of the above, then $\mathcal{L}'_v = \mathrm{image}\{H^1(D_v/I_{v,\rho}, W_v)\} \subset H^1(D_v, W_v)$.*

Proof. We must show that in any of the cases listed $\mathcal{L}'_v$ is the subspace that cuts out the ramification conditions at v required for a deformation to lie in $\mathcal{D}_{\Sigma}^{\det}(A)$.

- Suppose $v \mid p$. A deformation of ρ that is nearly ordinary at v can be represented by a homomorphism ρ_A that when restricted to I_v preserves the line $L_{A,v} = L_v \oplus \epsilon L_v \subset V_A = V \oplus \epsilon V$. It is then evident that the associated cocycle $c(\rho_A)$, when restricted to I_v, takes its values in W_v^0,and so $h(\rho_A)_v \in \mathcal{L}_v$. Thus $\mathcal{L}'_v \subset \mathcal{L}_v$. The opposite inclusion is equally straightforward.
- For v in Σ we have $\mathcal{L}'_v = H^1(D_v, W_v) = \mathcal{L}_v$.
- If v is none of the above, the requirement that the homomorphism ρ_A have restricted ramification at v is that its restriction to I_v induces a homomorphism from $D_v/I_{v,\rho}$ to $\mathrm{GL}_2(E[\epsilon])$ and thus that the associated deformation is identified with an element of $H^1(D_v, W_v)$ in the image of $H^1(D_v/I_{v,\rho}, W_v)$.

□

Returning now to the proof of our proposition, we must show that $\mathcal{L}'_v = \mathcal{L}_v$ for $v \not| p$ and $v \notin \Sigma$. We have

$$\mathcal{L}_v = \mathrm{image}\{H^1(D_v/I_v, W_v^{I_v})\} \subset \mathcal{L}'_v = \mathrm{image}\{H^1(D_v/I_{v,\rho}, W_v)\} \subset H^1(D_v, W_v).$$

But by Lemma 2.7, $\mathcal{L}_v = H^1(D_v, W_v)$; thus all the groups displayed in the line above are equal.

If the representation ρ restricted to I_v for v not dividing p factors through a finite quotient group, then independence of Σ follows from Corollary 2.9. □

2.3 Infinitesimal Hodge-Tate weights

Recall that p is completely split in K, and therefore, $D_v = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. By local class field theory, the Artin map gives us a homomorphism $\mathbf{Z}_p^* \hookrightarrow \mathbf{Q}_p^* \hookrightarrow D_v^{\mathrm{ab}}$, the latter group being the maximal abelian quotient of D_v. For $v|p$, we have that $\mathcal{L}_v \simeq H^1(D_v, W_v^0)$ by Lemma 2.4. Consider the one-dimensional E-vector space $\mathcal{E} := \mathrm{Hom}(\mathbf{Z}_p^*, E)$, and form the E-linear functional $\pi_v : \mathcal{L}_v \to \mathcal{E}$ given by composition

$$\mathcal{L}_v \simeq H^1(D_v, W_v^0) \xrightarrow{\epsilon} H^1(D_v, E) = \mathrm{Hom}(D_v^{\mathrm{ab}}, E) \to \mathrm{Hom}(\mathbf{Z}_p^*, E) = \mathcal{E}.$$

(in the above discussion, Hom always means continuous homomorphisms).

Definition 2.13. *Let $c \in H^1_\Sigma(K, W)$. For $v|p$, the* **infinitesimal Hodge-Tate weight of** c **at** v *is the image of c under the composition*

$$\omega_v : H^1_\Sigma(K, W) \longrightarrow \mathcal{L}_v \xrightarrow{\pi_v} \mathcal{E}$$

Putting the ω_v's together, we get the mapping of infinitesimal Selmer to infinitesimal weight space,

$$\omega := \bigoplus_{v|p} \omega_v : H^1_\Sigma(K, W) \to \bigoplus_{v|p} \mathcal{E} = \{\bigoplus_{v|p} \mathbf{Q}\} \otimes_{\mathbf{Q}} \mathcal{E}.$$

A vector w in this latter vector space $\{\bigoplus_{v|p} \mathbf{Q}\} \otimes_{\mathbf{Q}} \mathcal{E}$ will be called *rational* if w is expressible as $r \otimes e$ for $r \in \{\bigoplus_{v|p} \mathbf{Q}\}$ and $e \in \mathcal{E}$.

2.4 Motivations

One of the motivating questions investigated in this article is the following: under what conditions does the image of ω contain non-zero rational vectors?

If K is a totally real field and ρ is potentially semistable and odd at all real places of K, then one expects that ρ is modular. If ρ is modular, and K is totally real, the ordinary Hida families associated to ρ produce many deformations in which the Hodge-Tate weights vary rationally with respect to one another. We conjecture that in a strong sense the converse is almost true. Namely, that any deformation $c \in H^1_\Sigma(K, W)$ with $\omega(c)$ rational *only* occurs if c has arisen from the Hida family of some odd representation over a totally real field, the only exception being when c arises by induction from a family of representations over a CM field, or — in a final exceptional case that only conjecturally occurs when ρ is an Artin representation — when ρ arises by base change from some *even* representation.

We will be paying special attention in this article to *Artin representations* ρ; that is, to representations with finite image in $\mathrm{GL}_2(E)$. The results we obtain are, for the most part, *conditional* in that they depend on a certain ("natural," it seems to us) extension of the classical conjecture of Leopoldt.

Definition 2.14. *The representation ρ admits* infinitesimally classical deformations *if there exists a non-zero class $c \in H^1_\Sigma(K, W)$ with rational infinitesimal Hodge-Tate weights.*

Remark: If ρ is an Artin representation then by Corollary 2.9 there is an isomorphism $H^1_\Sigma(K,W) = H^1_\emptyset(K,W)$, so it suffices to consider the case $\Sigma = \emptyset$. If ρ is classical with distinct Hodge-Tate weights, then one also expects that all infinitesimal deformations of ρ are minimally ramified, and hence that $H^1_\Sigma(K,W) = H^1_\emptyset(K,W)$. This expectation arises from the automorphic setting, where an intersection of two Hida families in regular weight would contradict the semi-simplicity of the Hecke action.

3 The Leopoldt conjecture

3.1 Preliminaries

In this section we will let p be a prime number and $E/\mathbf{Q}_p$ a field extension, but some of what we say will have an analog if we take p to be ∞, i.e., if we consider our field E to be an extension field of $\mathbf{R}$.

Let $M/\mathbf{Q}$ be a finite Galois extension with Galois group G of order n. Denote by $\mathcal{O}_M^*$ group of units in the ring of integers of M, and let

$$U_{\text{Global}} := \mathcal{O}_M^* \otimes_{\mathbf{Z}} E$$

be the E-vector space obtained by tensor product with E. By the Dirichlet unit theorem, U_{Global} is of dimension either $n-1$ or $\frac{n}{2}-1$, depending upon whether M is totally real or totally complex. We will think of U_{Global} as a (left) $E[G]$-module induced by the natural action of G. In the sequel, our modules over non-commutative rings will be understood to be *left* modules unless we explicitly say otherwise (and, on occasion, we will say otherwise).

For v a place of M dividing p, let M_v denote the completion of M at v, and let $\mathcal{O}_{M_v}^* \subset M_v^*$ be the group of units in the ring of integers of the discrete valued field M_v. View

$$\widehat{\mathcal{O}}_{M_v}^* := \lim_{\leftarrow} \mathcal{O}_{M_v}^*/(\mathcal{O}_{M_v}^*)^{p^n},$$

the p-adic completion of $\mathcal{O}_{M_v}^*$, as $\mathbf{Z}_p$-module, and form

$$U_v := \widehat{\mathcal{O}}_{M_v}^* \otimes_{\mathbf{Z}_p} E,$$

which we think of as D_v-representation space, where $D_v \subset G$ is the decomposition group at v.

We have that U_v is a free $E[D_v]$-module of rank one. The group G acts transitively on the set of places $v|p$ and, correspondingly, conjugation by G is a transitive action on the set of decomposition groups $\{D_v\}_{v|p}$. Also, if $g \in G$ and $v|p$, the automorphism $g : M \to M$ induces an isomorphism also denoted $g : U_v \cong U_{gv}$.

Definition 3.1. $U_{\text{Local}} := \prod_{v|p} U_v$.

We view U_{Local} in the natural way as $E[G]$-module, the action of an element $g \in G$ sending the vector $(\dots, x_v, \dots)$ in $\prod_{v|p} U_v$ to $(\dots, y_v, \dots)$ where $y_{gv} = gx_v$ for $v|p$.

Fixing a place $w|p$ and putting $D = D_w$, we have natural identifications

$$\text{Ind}_D^G U_w = E[G] \otimes_{E[D]} U_w \cong \prod_{v|p} U_v = U_{\text{Local}}$$

by the homomorphism induced from the rule $g \otimes x \mapsto (\dots, x_v, \dots) \in \prod_{v|p} U_v$, where, for $g \in G$ and $x \in U_w$, we have that x_v (the v-th entry of the image) is 0 if $v \neq gw$ and is gx if $v = gw$. The $E[G]$-module U_{Local} is free of rank one.

Now let us consider "fields of definition" of the structure inherent to the $E[G]$-module U_{Local}. Let $F \subset E$ be a subfield (and we will be interested in the cases where F is a field of algebraic numbers).

If we choose a generator $u \in U_w$ of the $E[D]$-module U_w, then setting $U_{w,F} := F[D] \cdot u \subset U_w$ we see that $U_{w,F}$ is a free $F[D]$-module of rank one such that $U_{w,F} \otimes_F E \simeq U_w$. Refer to such a sub-$F[D]$-module of U_w as an *F-structure* on U_w.

Given any such F-structure on U_w we have a canonical right $F[D]$-module structure on $U_{w,F}$ as well. Namely, if

$$x = a \cdot u \in U_{w,F}$$

for $a \in F[D]$, and if $b \in F[D]$, the right-action of b on x is given by $x \cdot b := abu$. This extends by base change to give a right action of $E[D]$ on U_w. Given an F-structure on U_w, we get a corresponding F-structure on $U_{\text{Local}} = E[G] \otimes_{E[D]} U_w$ by setting

$$U_{\text{Local},F} := F[G] \otimes_{F[D]} U_{w,F}$$

which we view, then, as a (left) $F[G]$-module and a right $F[D]$-module.

These F-structures will be called *preferred F-structures on* U_{Local} although, in fact, we will consider no other ones, and we identify U_{Local} with $U_{\text{Local},F} \otimes_F E$, viewing it as a (left) $E[G]$-module and a right $E[D]$-module. The ambiguity in the choice of F-structure then boils down to which $(F[D])^*$-coset of generator of $E[D]$-module U_w we have chosen.

Suppose, then, we have such a preferred F-structure given on U_{Local}. By an **F-rational, right $E[D]$-submodule** $V \subset U_{\text{Local}}$ we mean a right $E[D]$-submodule of U_{Local} for which there exists a right $F[D]$-submodule $V_F \subset U_{\text{Local},F}$ such that $V = V_F \otimes_F E$. (For a systematic treatment of these issues, see the Appendix: section A.)

For any $v|p$, the natural injection $M \hookrightarrow M_v$ induces a homomorphism of $E[D_v]$-modules

$$U_{\text{Global}} \xrightarrow{\iota_v} U_v$$

and the product of these,

$$\iota = \iota_p = \prod_{v|p} \iota_v : U_{\text{Global}} \longrightarrow \prod_{v|p} U_v = U_{\text{Local}}$$

is naturally a homomorphism of $E[G]$-modules. We can also think of ι as the natural $E[G]$-homomorphism

$$\iota : U_{\text{Global}} \to \text{Ind}_D^G U_w$$

induced from the $E[D]$-homomorphism $\iota_w : U_{\text{Global}} \to U_w$.

The classical p-adic Leopoldt conjecture is equivalent to the statement that ι is injective for p any prime. This has been proven when $M/\mathbf{Q}$ is an abelian Galois extension [4].

Remark: In the situation where $p = \infty$, we have a *somewhat* analogous homomorphism

$$\iota_\infty : U_{\text{Global}} = \mathcal{O}_M^* \otimes \mathbf{R} \longrightarrow \prod_{v \text{ infinite}} \mathbf{R}$$

given by $\lambda(u) =$ the vector $(\dots, \lambda_v(u), \dots) \in \prod_{v|\infty} \mathbf{R}$ with $\lambda_v(u) = \log |u|_v$ for v real, and given by $\lambda_v(u) = 2\log |u|_v$ for v complex, which has been known (for over a century) to be injective.

3.2 The Strong Leopoldt Conjecture

Assume the classical Leopoldt Conjecture. The stronger version of Leopoldt's Conjecture that we shall formulate has to do with the manner in which the $E[G]$-submodule

$$\iota(U_{\text{Global}}) \subset U_{\text{Local}}$$

is skew to the F-structures we have considered on U_{Local}.

Definition 3.2 (Strong Leopoldt Condition for E/F)**.** *Let us say that $M/\mathbf{Q}$ satisfies the* **Strong Leopoldt Condition for** *E/F if the map ι is injective, and furthermore for any preferred F-structure on U_{Local}, and every F-rational right $E[D]$-submodule $Z \subset U_{\text{Local}}$, and every $E[G]$-submodule $Y \subset U_{\text{Local}}$ such that Y is isomorphic to U_{Global} as $E[G]$-module, we have the inequality of dimensions*

$$\dim(\iota(U_{\text{Global}}) \cap Z) \;\leq\; \dim(Y \cap Z).$$

For ease of reference, let us refer to the $E[G]$-submodules $Y \subset U_{\text{Local}}$ that appear in this definition as the **competitors** (i.e., to U_{Global}) and the F-rational right $E[D]$-submodules $Z \subset U_{\text{Local}}$ as the **tests**. So the Strong Leopoldt Condition for E/F is satisfied if, so to speak, U_{Global} "wins every test with every competitor," or at least manages a draw.

Conjecture 3.3 (Strong Leopoldt). *Let $E/\mathbf{Q}_p$ be any algebraic extension field, and $F \subset E$ any subfield where $F/\mathbf{Q}$ is algebraic. Let $M/\mathbf{Q}$ be any finite Galois extension. Then $M/\mathbf{Q}$ satisfies the Strong Leopoldt Condition for E/F.*

Remarks:

1. This conjecture is visibly a combination of the classical Leopoldt Conjecture and, as it seems to us, a fairly natural (genericity) hypothesis: the conjecture predicts that the E-vector space generated by global units — as it sits in the E-vector space of p-adic local units — is as *generic* as its module structure allows, relative to the preferred F-structures on the bi-module U_{Local} of p-adic local units. This genericity condition is discussed in a general context in the Appendix (compare definition A.8).
2. Note that the $\mathbf{Z}[G]$-module $\mathcal{O}_M^*$, or its torsion-free quotient, the *lattice* of global units, does *not* play a role in this Strong Leopoldt Condition, a condition that only concerns itself with the tensor product of this unit lattice with the large field E.
3. Choosing model F-representations V_η of G, for η running through the distinct irreducible F-representations of G, we have a canonical decomposition of the bi-module $U_{\text{Local},F}$ as

$$U_{\text{Local},F} = \bigoplus_{\eta} V_\eta \otimes V_\eta^{\#}$$

where for each η, $V_\eta^{\#} := \mathrm{Hom}_{F[G]}(V_\eta, U_{\text{Local},F})$ is viewed as a right $F[D]$-module. Put $V_{\eta,E} := V_\eta \otimes_F E$, and similarly $V_{\eta,E}^{\#} := V_\eta^{\#} \otimes_F E$. Assuming the classical Leopoldt conjecture we may identify $U_{\rm Global}$ with its image under ι_p in $U_{\rm Local}$ and write the image of $U_{\rm Global}$ in $U_{\rm Local}$ as a direct sum

$$U_{\rm Global} = \bigoplus_\eta V_\eta \otimes V^{\#}_{\eta,{\rm Global}} \subset \bigoplus_\eta V_\eta \otimes V^{\#}_{\eta,E},$$

the "placement" of $U_{\rm Global}$ in $U_{\rm Local}$ being determined by giving the vector subspaces

$$V^{\#}_{\eta,{\rm Global}} \subset V^{\#}_{\eta,E}$$

for each η. The Strong Leopoldt Conjecture, then, requires the $E[D]$-submodule $V^{\#}_{\eta,{\rm Global}}$ to be generically positioned (in the sense described above) with respect to the $F[D]$-structure of $V^{\#}_{\eta,E}$. In the special case of Galois extensions $M/\mathbf{Q}$ where the classical Leopoldt conjecture holds and, for each η, we have that either $V^{\#}_{\eta,{\rm Global}} = 0$ or $V^{\#}_{\eta,{\rm Global}} = V^{\#}_\eta$ (i.e., where $U_{\rm Global}$ consists of a direct sum of isotypic components of the $E[G]$-representation $U_{\rm Local}$) the Strong Leopoldt Condition (trivially) holds because — in this case — there are no competitors Y (in the sense we described above) other than $U_{\rm Global}$ itself.

Corollary 3.4. *If $M/\mathbf{Q}$ is an abelian extension, the Strong Leopoldt Conjecture holds.*

Proof. The classical Leopoldt conjecture holds for abelian extensions of $\mathbf{Q}$, and $U_{\rm Global}$ consists of a direct sum of isotypic components of the $E[G]$-representation $U_{\rm Local}$. □

Corollary 3.5. *If $M/\mathbf{Q}$ is a totally real Galois extension for which the classical Leopoldt conjecture holds, the Strong Leopoldt Conjecture holds.*

Proof. Since M is totally real, Leopoldt's conjecture is equivalent to the statement that there is an isomorphism $U_{\rm Local}/U_{\rm Global} \simeq E$ as G-modules. Equivalently, $U_{\rm Global}$ consists of a direct sum of all the non-trivial isotypic components of $U_{\rm Local}$ as an $E[G]$-representation.

□

The above are cases, then, where (given the classical conjecture) the Strong Leopoldt Conjecture holds vacuously.

3.3 The Strong Leopoldt Conjecture and the p-adic Schanuel conjecture

To get a sense of the nature of the Strong Leopoldt conjecture in a relatively simple situation, but a more interesting one than treated in Corollaries 3.4 and 3.5, let us suppose that:

- the decomposition group D is trivial,
- the classical Leopoldt conjecture holds, and
- There is an irreducible G-representation η, and a nonzero vector $v^{\#} \in V^{\#}_{\eta,E}$ such that the $E[G]$-sub-representation $U_{\rm Global} \subset U_{\rm Local}$ is a direct sum

$$U_{\rm Global} \quad = \quad I \bigoplus \{V_{\eta,E} \otimes_E v^{\#}\} \ \subset \ U_{\rm Local}$$

where I is an $E[G]$-representation that is a direct sum of *some* isotypic components in $U_{\rm Local}$.

(To make it interesting, of course, we would want that this irreducible representation η to be of dimension > 1. We shall consider some explicit examples when this occurs in subsequent sections.) In the above situation the only possible *competitors* Y are of the form

$$Y \;\; = \;\; I \bigoplus \{V_{\eta,E} \otimes_E y^{\#}\} \;\subset\; U_{\text{Local}}$$

for some nonzero vector $y^{\#} \in V^{\#}_{\eta,E}$. One easily sees that — in this situation — to test whether or not the Strong Leopoldt Conjecture holds, it suffices to consider only *test vector spaces* Z contained in $V_\eta \otimes V^{\#}_\eta$. Explicitly, the Strong Leopoldt Conjecture will hold if and only if

$$\dim\left(\{V_{\eta,E} \otimes_E v^{\#}\} \cap Z_E\right) \;\leq\; \dim\left(\{V_{\eta,E} \otimes_E y^{\#}\} \cap Z_E\right)$$

for all F-vector subspaces $Z \subset V_\eta \otimes V^{\#}_\eta$. Let $d := \dim_F(V^{\#}_\eta)$ and choose an F-basis for $V^{\#}_\eta$, identifying $V^{\#}_\eta$ with F^d, and (tensoring with E) $V^{\#}_{\eta,E}$ with E^d. After this identification we may view the vector $v^{\#}$ as a d-tuple of elements of E

$$v^{\#} := (e_1, e_2, \ldots, e_d).$$

Without loss of generality we may assume that $e_d = 1$.

Theorem 3.6. *In the situation described above,*

1. *If the Strong Leopoldt Condition for E/F holds, then the elements $e_1, e_2, \ldots, e_d = 1 \in E$ are linearly independent over F.*
2. *If the transcendence degree of the field $F(e_1, e_2, \ldots, e_{d-1}) \subset E$ is $d-1$, then the Strong Leopoldt Condition for E/F holds in this situation.*

Proof. We begin with part one. If $e_1, e_2, \ldots, e_d \in E$ satisfies a nontrivial F-linear relation explicitly given by $\lambda(e_1, e_2, \ldots, e_d) = 0$, let $V^{\#}_o \subset V^{\#}_\eta$ be the kernel of λ; so $v^{\#} \in V^{\#}_o$. Define the F-vector test subspace

$$Z := V_\eta \otimes_F V^{\#}_o \subset V_\eta \otimes_F V^{\#}_\eta,$$

and choose as "competitor" the $E[G]$-sub-representation

$$Y := V_{\eta,E} \otimes_E y^{\#} \subset V_{\eta,E} \otimes_E V^{\#}_{\eta,E}.$$

where $y^{\#} \in V^{\#}_{\eta,E}$ is any vector such that $\lambda(y^{\#}) \neq 0$. Then

$$1 = \dim(\{V_{\eta,E} \otimes_E v^{\#}\} \cap Z_E) \;>\; \dim(Y \cap Z_E) = 0.$$

This proves part one.

Suppose now that the transcendence degree of the field $F(e_1, e_2, \ldots, e_{d-1}) \subset E$ is $d-1$. Form the polynomial ring $R := F[X_1, X_2, \ldots, X_{d-1}]$, and consider the R-modules $V_{\eta,R} := V_\eta \otimes_F R$, $V^{\#}_{\eta,R} := V^{\#}_\eta \otimes_F R = R^d$, and $Z_R = Z \otimes_F R \subset V_{\eta,R} \otimes_R V^{\#}_{\eta,R}$. Now consider the element $X^{\#} := (X_1, X_2, \ldots, X_{d-1}, 1) \in R^d = V^{\#}_{\eta,R}$ and form the R-submodule

$$\{V_{\eta,R} \otimes_R X^{\#}\} \bigcap Z_R \;\subset\; V_{\eta,R} \otimes_R V^{\#}_{\eta,R}.$$

By sending the independent variables $(X_1, X_2, \ldots, X_{d-1})$ to $(e_1, e_2, \ldots, e_{d-1})$ we get an injective F-algebra homomorphism from R to E identifying the base change to E of the R-module $\{V_{\eta,R} \otimes_R X^{\#}\} \bigcap Z_R$ with the E-vector space $\{V_{\eta,E} \otimes_E v^{\#}\} \cap Z_E$. For any $y^{\#} = (c_1, c_2, \ldots, c_{d-1}, 1) \in E^d$, by sending the independent variables $(X_1, X_2, \ldots, X_{d-1})$ to $(c_1, c_2, \ldots, c_{d-1})$ we get an F-algebra homomorphism from R to E identifying the base change to E of the R-module $\{V_{\eta,R} \otimes_R X^{\#}\} \bigcap Z_R$ with the E-vector space

$$Y \cap Z_E = \{V_{\eta,E} \otimes_E y^{\#}\} \cap Z_E.$$

An appeal to the principle of upper-semi-continuity then guarantees that

$$\dim(\{V_{\eta,E} \otimes_E v^{\#}\} \cap Z_E) \leq \dim(\{V_{\eta,E} \otimes_E y^{\#}\} \cap Z_E)$$

concluding the proof of our theorem. □

The Schanuel Conjecture is an (as yet unproved) elegant statement regarding the transcendentality of the exponential function evaluated on algebraic numbers (or of the logarithm, depending upon the way it is formulated; we will give both versions below). Its proof would generalize substantially the famous Gelfond-Schneider Theorem that establishes the transcendentality of α^{β} for algebraic numbers α and β (other than the evidently inappropriate cases $\alpha = 0, 1$ or β rational). There is a p-adic analog to the Schanuel Conjecture, currently unproved, and a p-adic analog to the Gelfond-Schneider Theorem established by Kurt Mahler.

Here, then, are the full-strength conjectures in two, equivalent, formats.

Conjecture 3.7 (**Schanuel** — exponential formulation)**.** *Given any n complex numbers $z_1, \ldots, z_n$ which are linearly independent over $\mathbf{Q}$, the extension field $\mathbf{Q}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n))$ has transcendence degree at least n over $\mathbf{Q}$.*

Conjecture 3.8 (**Schanuel** — logarithmic formulation)**.** *Given any n nonzero complex numbers $z_1, \ldots, z_n$ whose logs (for any choice of the multi-valued logarithm) are linearly independent over the rational numbers $\mathbf{Q}$, then the extension field $\mathbf{Q}(\log z_1, \ldots, \log z_n, z_1, \ldots, z_n)$ has transcendence degree at least n over $\mathbf{Q}$.*

A somewhat weaker formulation of this conjecture is as follows:

Conjecture 3.9 (**Weak Schanuel** — logarithmic formulation)**.** *Given any n algebraic numbers $\alpha_1, \ldots, \alpha_n$ whose logs are linearly independent over the rational numbers $\mathbf{Q}$, then the extension field $\mathbf{Q}(\log \alpha_1, \ldots, \log \alpha_n)$ has transcendence degree n over $\mathbf{Q}$.*

We are interested in the p-adic version of the above conjecture, namely:

Conjecture 3.10 (**Weak Schanuel** — p-adic logarithmic formulation)**.** *Let $\alpha_1, \ldots, \alpha_n$ be n nonzero algebraic numbers contained in a finite extension field E of $\mathbf{Q}_p$. Let $\log_p : E^* \to E$ be the p-adic logarithm normalized so that $\log_p(p) = 0$. If $\log_p \alpha_1, \ldots, \log_p \alpha_n$ are linearly independent over the rational numbers $\mathbf{Q}$, then the extension field $\mathbf{Q}(\log \alpha_1, \ldots, \log \alpha_n) \subset E$ has transcendence degree n over $\mathbf{Q}$.*

Theorem 3.11. *If $M/\mathbf{Q}$ satisfies the hypotheses of the beginning of this subsection, then the classical Leopoldt Conjecture plus the weak p-adic Schanuel Conjecture implies the Strong Leopoldt Conjecture.*

Proof. This follows directly from part 2 of Theorem 3.6. □

In the next two subsections we examine cases of the above set-up.

3.4 The Strong Leopoldt Condition for complex S_3-extensions of $\mathbf{Q}$

Consider the case where $M/\mathbf{Q}$ is Galois with $G = S_3$, the symmetric group on three letters. Suppose that p splits completely in M, so the decomposition group D is trivial, and M is totally complex. Let E be an algebraic closure of $\mathbf{Q}_p$, and $F \subset E$ an algebraic extension of $\mathbf{Q}$ contained in E.

Theorem 3.12. *The Strong Leopoldt Condition is true in this situation.*

We begin our discussion assuming the hypotheses and notation of the opening paragraph of this subsection. We have a canonical isomorphism of $F[G]$-modules

$$U_{\mathrm{Local},F} \simeq F \oplus F \oplus V_\eta \otimes_F V_\eta^\#,$$

where the action of G on the various summands is as follows: the action on the first summand is trivial, the action on the second is via the sign representation, the action on the third is via action on the first tensor factor, i.e., $g \cdot (x \otimes x^\#) = (gx) \otimes x^\#$ for $g \in G$, $x \in V_\eta$ and $x^\# \in V_\eta^\#$), where V_η is the irreducible representation of $G = S_3$ on a two-dimensional vector space over F. The $E[G]$-module U_{Global} is isomorphic to $V_{\eta,E} := V_\eta \otimes_F E$.

The homomorphism $\iota_p : U_{\mathrm{Global}} \to U_{\mathrm{Local}}$ is injective (i.e., the classical Leopoldt Conjecture is — in fact, trivially — true) since the G-representation U_{Global} is irreducible and ι_p doesn't vanish identically. Moreover, the image of ι_p is necessarily contained in the third summand of the decomposition displayed above; i.e., we may view ι_p as being an injection of $E[G]$-modules

$$\iota_p : U_{\mathrm{Global}} \to V_{\eta,E} \otimes_E V_{\eta,E}^\# = (V_\eta \otimes_F V_\eta^\#) \otimes_F E \subset U_{\mathrm{Local}},$$

identifying U_{Global} with the $E[G]$-sub-representation $V_\eta \otimes_F v^\# \subset V_{\eta,E} \otimes_E V_{\eta,E}^\#$ for a nonzero vector $v^\# \in V_{\eta,E}^\#$. In particular, we are in exactly the situation discussed in subsection 3.3.

Choose an F-basis $\{v_1^\#, v_2^\#\}$ of $V_\eta^\#$ and write $v^\# = a_1 v_1^\# + a_2 v_2^\#$ for specific elements $a_1, a_2 \in E$ we see that the subset of $E \cup \{\infty\}$ consisting in the set of images of $\mu := a_1/a_2$ under all linear fractional transformations with coefficients in F (call this the subset of F-**slopes** of U_{Global}) is an invariant of our situation.

Lemma 3.13. *In the situation described above:*

1. *If the Strong Leopold condition holds for E/F, then the set of F-slopes of λ is not contained in F.*
2. *If the set of F-slopes of λ is not contained in any quadratic extension of F, then the Strong Leopold condition holds for E/F.*

Proof. If the set of F-slopes of U_{Global} is contained in F, then the image of U_{Global} in U_{Local} is itself the base change to E of an $F[G]$-sub-representation in the preferred F-structure of U_{Local} which is an F-vector subspace (relative to the preferred F-structure) of U_{Local}, and taking this $F[G]$-sub-representation itself as test object Z — and $Y \subset U_{\text{Local}}$ as *any* irreducible two-dimensional $E[G]$ sub-representation different from U_{Global} — give the dimension inequality

$$\dim(U_{\text{Global}} \cap Z) > \dim(Y \cap Z)$$

that contradicts the Strong Leopoldt Condition. This proves part one.

Suppose now, that the set of F-slopes of λ is not contained in in any quadratic extension of F. As discussed in Remark 3 of the previous subsection, the $E[G]$-representation subspaces $Y \subset U_{\text{Local}}$ that are isomorphic to U_{Global} (as $E[G]$-representations) are of the form $V_{\eta,E} \otimes_E y^{\#}$ for some nonzero vector $y^{\#} \in V^{\#}_{\eta,E}$. Let $Z \subset V_\eta \otimes_F V^{\#}_\eta$ be a test F-rational vector space, and let $Z_E = Z \otimes_F E$. The three cases that are relevant to us are when $\dim(Z_E) = 1, 2, 3$. One checks that (since the set of F-slopes of U_{Global} is not contained in F) $V_{\eta,E} \otimes_E v^{\#}$ does not contain an F-rational line, nor is annihilated by a nontrivial F-rational linear form. Therefore we need only treat the case where Z_E is of dimension two. Now an inequality of dimensions of the form $\dim(Y \cap Z_E) < \dim(U_{\text{Global}} \cap Z_E)$ can only happen if $\dim(Y \cap Z_E) = 0$ and $\dim(U_{\text{Global}} \cap Z_E) = 1$, i.e., if $Y + Z_E = (V_{\eta,E} \otimes_E V^{\#}_{\eta,E})$ while $U_{\text{Global}} + Z_E$ is a proper subspace of $(V_{\eta,E} \otimes_F V^{\#}_{\eta,E})$.

One can now make this explicit in terms of 4×4 matrices as follows. We can choose an F-bases $\{v_1, v_2\}$ of the vector space V_η and $\{v_1^{\#}, v_2^{\#}\}$ of the vector space $V_\eta^{\#}$ so that we may write $v^{\#} = v_1^{\#} + \mu v_2^{\#}$ and $y^{\#} = v_1^{\#} + \nu v_2^{\#}$ where by hypothesis μ is not contained in any quadratic extension of F (and we know nothing about ν). We have a basis $\{v_{i,j} := v_i \otimes v_j^{\#}\}_{i,j}$ for $V_{\eta,E} \otimes_E V^{\#}_{\eta,E}$ and in terms of this basis:

- U_{Global} is generated by $v_{1,1} + \mu v_{1,2}$ and $v_{2,1} + \mu v_{2,2}$,
- any competitor Y is generated by $v_{1,1} + \nu v_{1,2}$ and $v_{2,1} + \nu v_{2,2}$,
- any test Z is generated by $\sum_{i,j} a_{i,j} v_{i,j}$ and $\sum_{i,j} b_{i,j} v_{i,j}$ for $a_{i,j}, b_{i,j} \in F$.

We leave it to the reader to check (a) that if μ is not contained in any quadratic extension field of F the rank of the matrix

$$\begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 0 & 1 & \mu \\ a_{1,1} & a_{1,2} & a_{2,1} & a_{2,2} \\ b_{1,1} & b_{1,2} & b_{2,1} & b_{2,2} \end{pmatrix}$$

is greater than or equal to the rank of the matrix

$$\begin{pmatrix} 1 & \nu & 0 & 0 \\ 0 & 0 & 1 & \nu \\ a_{1,1} & a_{1,2} & a_{2,1} & a_{2,2} \\ b_{1,1} & b_{1,2} & b_{2,1} & b_{2,2} \end{pmatrix}$$

for any ν, and (b) that this concludes the proof of our lemma. $\square$

3.5 Proof of Theorem 3.12

Given Lemma 3.13 it clearly suffices to show that the slope μ defined in the previous subsection is transcendental. Here, more concretely, is how to see this invariant μ. If $K_1, K_2, K_3 \subset M$ are the three conjugate subfields of order three, let $\epsilon_1, \epsilon_2, \epsilon_3$ be fundamental units of these subfields ($\epsilon_i > 1$ in the unique real embedding $K_i \hookrightarrow \mathbf{R}$) and note that $\epsilon_1\epsilon_2\epsilon_3 = 1$, and that the group G acts as a full group of permutations of the set $\{\epsilon_1, \epsilon_2, \epsilon_3\}$.

Fix a place w of M over p and let $\mathcal{O}_w = \mathbf{Z}_p$ denote the ring of integers in the w-adic completion of M. Let $\epsilon_{1,w}, \epsilon_{2,w}, \epsilon_{3,w} \in \mathcal{O}_w^* = \mathbf{Z}_p^*$ denote the images of the units $\epsilon_1, \epsilon_2, \epsilon_3$. Let $\log_p : \mathbf{Z}_p^* \to \mathbf{Q}_p$ denote the classical p-adic logarithm. The F-slope of U_{Global} in U_{Local} is, up to F-linear fractional transformation, equal to

$$\mu = \log_p(\epsilon_{1,w})/\log_p(\epsilon_{2,w}).$$

The transcendence of $\log_p(\epsilon_{1,w})/\log_p(\epsilon_{2,w})$ follows from Mahler's p-adic version of the Gelfond-Schneider theorem ([25], Hauptsatz, p.275). Mahler proved that for any two algebraic numbers α, β in $\overline{\mathbf{Q}}_p$, either $\log_p(\alpha)/\log_p(\beta)$ is rational or transcendental. Since ϵ_1 and ϵ_2 are linearly independent in the torsion-free quotient of $\mathcal{O}_M^*$, the ratio $\log_p(\alpha)/\log_p(\beta)$ is not rational, and hence is transcendental.

Remark: The Archimedean analog of the above transcendence statement follows from the classical result of Gelfond-Schneider. Specifically, embed M in $\mathbf{C}$ (in one of the three possible ways). For $i \neq j$ let $\log(\epsilon_i)/\log(\epsilon_j)$ refer to the ratios of any of the values of the natural logarithms of ϵ_i and of ϵ_j; these ratios are transcendental. A formally similar question concerning p-adic transcendence in the arithmetic of elliptic curves is — to our knowledge — currently unknown. Namely, let $M/\mathbf{Q}$ be an S_3-extension, $\sigma, \tau \in S_3$ denoting the elements of order 2 and 3 respectively. Let v be a place of M of degree one dividing a prime number p. Now let A be an elliptic curve over $\mathbf{Q}$ with Mordell-Weil group over M denoted $A(M)$ and suppose that the S_3 representation space $A(M) \otimes \mathbf{Q}_p$ is the two-dimensional irreducible representation of S_3. We have the natural homomorphism $x \mapsto x_v$ of $A(M) \otimes \mathbf{Q}_p$ onto the one-dimensional $\mathbf{Q}_p$-vector space $A(M_v) \otimes \mathbf{Q}_p$. Let $\alpha \in A(M)/\mathrm{torsion} \subset A(M) \otimes \mathbf{Q}_p$ be a nontrivial element fixed by σ and let $\beta := \tau(\alpha)$. Neither of the elements $\alpha_v, \beta_v \in A(M) \otimes \mathbf{Q}_p$ vanish. We can therefore take the ratio $\alpha_v/\beta_v \in \mathbf{Q}_p$, this ratio being independent of the initial choice of α (subject to the conditions we imposed). Is this ratio α_v/β_v a transcendental p-adic number?

3.6 The Strong Leopoldt Condition for complex A_4-extensions of Q

One gets a slightly different slant on some features of the Strong Leopoldt Condition if one considers the case where $M/\mathbf{Q}$ is Galois with $G = A_4$, the alternating group on four letters. Suppose, again, that p splits completely in M, so the decomposition group D is trivial, and suppose that M is totally complex. Let E be an algebraic closure of $\mathbf{Q}_p$, and $F \subset E$ an algebraic extension of $\mathbf{Q}$ contained in E. Given a preferred F-structure, the $E[G]$-module U_{Local} then admits an isomorphism $U_{\mathrm{Local}} \simeq F[G] \otimes_F E$, and any change of preferred F-structure amounts to a scalar shift — multiplication by a nonzero element of E. We have a canonical isomorphism of $F[G]$-modules

$$U_{\mathrm{Local},F} \simeq F \oplus F \oplus F \oplus V_\eta \otimes_F V_\eta^{\#},$$

where the action of G on the first three summands is via the three one-dimensional representations of A_4, while the fourth summand, $V_\eta \otimes_F V_\eta^{\#}$, is the isotypic component of the (unique) three-dimensional representation V_η; so $V_\eta^{\#}$ is again of dimension 3. Put, as usual, $V_{\eta,E} := V_\eta \otimes_F E$. The $E[G]$-module $U_{\rm Global}$ is isomorphic to $E \oplus E \oplus V_{\eta,E}$ where the action of G on the first two summands is via the two nontrivial one-dimensional characters. Here too, the homomorphism $\iota_p : U_{\rm Global} \to U_{\rm Local}$ is injective (i.e., the classical Leopoldt Conjecture is true since the G-representation space $U_{\rm Global}$ contains only representations with multiplicity one). We thus have an injection $\iota_p : U_{\rm Global} \to U_{\rm Local}$ identifying $U_{\rm Global}$ with the direct sum

$$U_{\rm Global} \quad = \quad I \bigoplus \{V_{\eta,E} \otimes_E v^{\#}\} \ \subset \ U_{\rm Local},$$

with $v^{\#} \in V_{\eta,E}^{\#}$ as discussed in subsection 3.3.

Corollary 3.14. *Let $M/\mathbf{Q}$ be as in this subsection. The weak p-adic Schanuel Conjecture implies the Strong Leopoldt Conjecture for $M/\mathbf{Q}$ relative to E/F.*

Concretely, here is what is involved. Let K be any non-totally real extension of $\mathbf{Q}$ of degree 4 whose Galois closure $M/\mathbf{Q}$ is an A_4-field extension. Then for this field we have $r_1 = 0$ and $r_2 = 2$, so the rank of the group of units is 1. Let u be any unit in K that is not a root of unity. Since $u \in K \subset M$ is fixed by an element of order three in $\mathrm{Gal}(M/\mathbf{Q})$, it has precisely four conjugates $u = u_1, u_2, u_3, u_4 \in M$ (with $u_1u_2u_3u_4 = 1$). Let v be a place of M of degree one, and let p be the rational prime that v divides. By means of the embedding $M \subset M_v = \mathbf{Q}_p$ we view those four conjugates as elements of $\mathbf{Q}_p^*$ and consider the four p-adic numbers

$$\log_p(u_1), \log_p(u_2), \log_p(u_3), \log_p(u_4) \in \mathbf{Q}_p.$$

The sum of these four numbers is 0, and the weak p-adic Schanuel Conjecture applied to the numbers $\log_p(u_1), \log_p(u_2), \log_p(u_3)$ implies the Strong Leopoldt Condition by Theorem 3.11.

Remark. It would be interesting to work out the relationship between the weak p-adic Schanuel Conjecture and our Strong Leopoldt Conjecture, especially in some of the other situations where the Classical Leopoldt Conjecture has been proven. Specifically, consider the cases where $M/\mathbf{Q}$ is a (totally) complex Galois extension with Galois group G such that for every irreducible character χ of G we have the inequality

$$(\chi(1) + \chi(c))(\chi(1) + \chi(c) - 2) < 4\chi(1).$$

Then (cf. Corollary 2 of section 1.2 of [23]) the Classical Leopoldt Conjecture holds for the number field M. This inequality is satisfied, for example, for certain S_4-extensions, and $\mathrm{GL}_2(\mathbf{F}_3)$-extensions, of $\mathbf{Q}$. See also Theorem 7 of [34] and conjectures formulated by Damien Roy in [35]. We are thankful to Michel Waldschmidt and Damien Roy for very helpful correspondence regarding these issues.

4 Descending Group Representations

Let E be an algebraically closed field of characteristic zero. If V is a finite dimensional vector space over E, we will denote by PV the associated projective space over E. Denote by $\mathrm{GL}(V)$ the group $\mathrm{Aut}_E(V)$ of E-linear automorphisms of V, and by $\mathrm{PGL}(V)$ the group $\mathrm{Aut}_E(PV)$ of linear projective automorphisms of PV over E.

Let A be a group. The automorphism group $\mathrm{Aut}(A)$ of A acts on the set of homomorphisms of A to any group by the standard rule: $\iota \circ \phi(a) := \phi \cdot \iota^{-1}(a)$.

4.1 Projective Representations

We start this section by considering *projective* representations of A on E-vector spaces; more precisely: equivalence classes Φ of homomorphisms

$$\phi : A \to \mathrm{PGL}(V)$$

for V a finite dimensional E-vector space, where equivalence means up to conjugation by an element in $\mathrm{PGL}(V)$. Note that the representation Φ need not necessarily lift to a (linear) representation of A on V.

The action of the automorphism group $\mathrm{Aut}(A)$ on projective representations of A factors through the quotient $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$, the group of *outer automorphisms* of A.

Definition 4.1. *If Φ is a projective representation of A, let $\mathrm{Aut}_\Phi(A) \subseteq \mathrm{Aut}(A)$ denote the subgroup of automorphisms of A that preserve Φ. Let $\mathrm{Out}_\Phi(A) \subseteq \mathrm{Out}(A)$ denote the image of $\mathrm{Aut}_\Phi(A)$.*

Lemma 4.2. *Let $\phi : A \to \mathrm{PGL}(V)$ be a homomorphism for which the associated projective representation Φ is irreducible. Then there is a unique homomorphism $\widetilde{\phi} : \mathrm{Aut}_\Phi(A) \to \mathrm{PGL}(V)$ such that the composition*

$$A \to \mathrm{Inn}(A) \to \mathrm{Aut}_\Phi(A) \xrightarrow{\widetilde{\phi}} \mathrm{PGL}(V).$$

is equal to ϕ.

Proof. If $\alpha \in \mathrm{Aut}_\Phi(A)$, there is a unique intertwining element $\widetilde{\phi}(\alpha) \in \mathrm{PGL}(V)$ such that $\alpha \circ \phi = \widetilde{\phi}(\alpha) \cdot \phi \cdot \widetilde{\phi}(\alpha)^{-1}$. Uniqueness comes from Schur's lemma, given that Φ is an irreducible projective representation. One checks that $\widetilde{\phi}$ is a homomorphism. If $a \in A$ then $\phi(a) \in \mathrm{PGL}(V)$ is "a" (hence "the") intertwiner for the automorphism given by conjugation by a, and hence $\widetilde{\phi}$ extends ϕ. ☐

Note that if Φ is an irreducible projective representation of A on PV then the center of A acts trivially on PV by Schur's lemma.

4.2 Linear Representations

In this section we consider some of the analogs of the results and constructions in section 4.1 for linear representations. By linear representations (over E) we mean equivalence classes Ψ of homomorphisms

$$\psi : A \to \mathrm{GL}(V).$$

The action of $\mathrm{Aut}(A)$ and $\mathrm{Out}(A)$ on linear representations of A is compatible with the operation that associates to any linear representation Ψ its corresponding projective representation Φ.

Definition 4.3. *If Ψ is a linear representation of A, let $\mathrm{Aut}_\Psi(A) \subseteq \mathrm{Aut}(A)$ denote the subgroup of automorphisms of A that preserve the associated linear representation Ψ. Let $\mathrm{Out}_\Psi(A) \subseteq \mathrm{Out}(A)$ denote the image of $\mathrm{Aut}_\Psi(A)$.*

Given a group B and a homomorphism $\phi : B \to \mathrm{PGL}(V)$, the pullback of ϕ with respect to the exact sequence

$$0 \to E^\times \to \mathrm{GL}(V) \to \mathrm{PGL}(V) \to 0$$

yields a central extension of B by $E^\times$, and therefore a "factor system," and, in particular, a well-defined class $\lambda(\phi) \in H^2(B, E^\times)$ whose vanishing is equivalent to the existence of a linear representation lifting the representation ϕ. If $A \subseteq B$ is a normal subgroup such that the homomorphism ϕ restricted to A lifts to a homomorphism $\psi : A \to \mathrm{GL}(V)$, then $\lambda(\phi)$ arises from a class in $H^2(B/A, E^\times)$.

Lemma 4.4. *Let A be a normal subgroup of B, and let Ψ be an irreducible linear representation of A. Suppose that the natural homomorphism $B \to \mathrm{Aut}(A)$ induced from conjugation has the property that its image lies in $\mathrm{Aut}_\Psi(A)$. Then if the quotient group B/A is finite cyclic, the linear representation Ψ of A on V extends to a representation $\widetilde{\Psi}$ of B on V unique up to a character of B/A.*

Proof. Let Φ denote the irreducible projective representation associated to Ψ, and let $\phi : A \to \mathrm{PGL}(V)$ be a homomorphism in the class of Φ. There is a natural inclusion of groups $\mathrm{Aut}_\Psi(A) \subseteq \mathrm{Aut}_\Phi(A)$. Thus, by Lemma 4.2, the map ϕ factors as the composition

$$A \to \mathrm{Inn}(A) \to \mathrm{Aut}_\Phi(A) \xrightarrow{\widetilde{\phi}} \mathrm{PGL}(V).$$

Composition of $B \to \mathrm{Aut}_\Psi(A) \subseteq \mathrm{Aut}_\Phi(A)$ with ϕ defines a map $\widetilde{\phi} : B \to \mathrm{PGL}(V)$. By Lemma 4.2 $\widetilde{\phi}$ extends ϕ. Suppose that B/A is finite cyclic of order n. Let $\sigma \in B$ project to a generator of B/A. We begin by defining $\widetilde{\psi}(\sigma) \in \mathrm{GL}(V)$ to be a lifting of $\widetilde{\phi}(\sigma) \in \mathrm{PGL}(V)$ with the property that $\widetilde{\psi}(\sigma)^n = \psi(\sigma^n)$. Such a choice is possible because E is algebraically closed. Moreover, such an equation is clearly necessary if $\widetilde{\psi}$ is to extend ψ; there are precisely n such choices, each differing by n-th roots of unity. Each element of B can be written uniquely in the form $\sigma^i x$ for $0 \le i < n$ and $x \in A$, and we define

$$\widetilde{\psi}(\sigma^i x) = \widetilde{\psi}(\sigma)^i \psi(x),$$

as we are forced to do if we wish $\widetilde{\psi}$ to be a homomorphism extending ψ. We first observe that the identity above holds for *all* i by induction, since

$$\widetilde{\psi}(\sigma^{i+n}x) = \widetilde{\psi}(\sigma^i\sigma^n x) = \widetilde{\psi}(\sigma)^i\psi(\sigma^n x) = \widetilde{\psi}(\sigma)^i\psi(\sigma^n)\psi(x) = \widetilde{\psi}(\sigma)^{i+n}\psi(x).$$

Sublemma 2. *There is an equality $\psi(\sigma^{-1}x\sigma) = \widetilde{\psi}(\sigma)^{-1}\psi(x)\widetilde{\psi}(\sigma)$.*

Proof. By assumption, conjugation of A by σ acts as an element of $\mathrm{Aut}_\Psi(A)$, and thus the homomorphisms ψ and ψ^σ (i.e., ψ conjugated by σ) are related by an intertwiner in $\mathrm{PGL}(V)$ unique up to scalar multiple. The same argument applied to the projectivization shows that ϕ and ϕ^σ (i.e., ϕ conjugated by σ) are related by a unique intertwiner, which is the projectivization of the corresponding linear one. Yet by construction $\widetilde{\psi}(\sigma)$ is the corresponding intertwiner, and thus the equality $\psi(\sigma^{-1}x\sigma) = \widetilde{\psi}(\sigma)^{-1}\psi(x)\widetilde{\psi}(\sigma)$ holds for *any* lift $\widetilde{\psi}(\sigma)$ of $\widetilde{\phi}(\sigma)$. ∎

To show that $\widetilde{\psi}$ is a homomorphism we are required to show that $\widetilde{\psi}(\sigma^p x \cdot \sigma^q y) = \widetilde{\psi}(\sigma^p x)\widetilde{\psi}(\sigma^q y)$. Now

$$\widetilde{\psi}(\sigma^p x \cdot \sigma^q y) = \widetilde{\psi}(\sigma^{p+q}(\sigma^{-q} x \sigma^q) y) = \widetilde{\psi}(\sigma)^{p+q} \psi(\sigma^{-q} x \sigma^q)\psi(y).$$

Applying Sublemma 2 q-times we may write this as

$$\widetilde{\psi}(\sigma)^{p+q}\widetilde{\psi}(\sigma)^{-q}\psi(x)\widetilde{\psi}(\sigma)^q\psi(y) = \widetilde{\psi}(\sigma)^p\psi(x)\widetilde{\psi}(\sigma)^q\psi(y) = \widetilde{\psi}(\sigma^p x)\widetilde{\psi}(\sigma^q y),$$

which was to be shown. To conclude the proof of Proposition 4.4 we note that tensoring the homomorphism $\widetilde{\psi}$ with the n distinct characters of B/A provides n distinct homomorphisms extending ψ; these must be all of them by the discussion above. $\square$

Let V be a two-dimensional E-vector space, Γ a finite group, and $\phi : \Gamma \hookrightarrow \mathrm{PGL}(V)$ an injective homomorphism such that the projective representation Φ that ϕ determines is irreducible (and visibly: faithful). Then Γ is isomorphic to one of the groups in the collection $\mathfrak{G} = \{D_n, n \text{ odd}, A_4, S_4, A_5\}$. Note that all these groups have trivial center.

If $W = \mathrm{Hom}'(V,V)$ denotes the hyperplane in $\mathrm{Hom}(V,V)$ consisting of endomorphisms of V of trace zero, then ϕ induces a *linear* faithful action of Γ on W.

Definition 4.5. *Let $X \subseteq W$ be the unique vector subspace stabilized by the above action of Γ on which Γ acts irreducibly and faithfully. Explicitly,*

1. *If $\Gamma \simeq A_4, S_4$, or A_5, then $W = X$.*
2. *If $\Gamma \simeq D_n$, n odd, then $W = X \oplus \epsilon$, where ϵ stands for the one-dimensional Γ representation given by the quadratic character of $\Gamma \simeq D_n$.*

Denote by Ψ the (irreducible, faithful, linear) representation of Γ on X.

With V a two-dimensional vector space as above, let us be given G a finite group, H a normal subgroup of G equipped with an irreducible projective representation Φ' of H on PV. Let $\mathrm{Ker} \subset H$ be the kernel of this projection representation Φ', so that Φ' factors through a *faithful* representation, Φ of $\Gamma := H/\mathrm{Ker}$ on PV. Therefore Γ and Φ are among the groups and representations classified in the discussion above; moreover, we have a corresponding linear representation Ψ of Γ on X, as described. Denote by Ψ' the linear representation of H obtained by composing Ψ with the surjective homomorphism $H \to \Gamma$.

Let $N(\mathrm{Ker})$ be the normalizer of Ker in G, so that we have the sequence of subgroups

$$\{1\} \subset \mathrm{Ker} \subset H \subset N(\mathrm{Ker}) \subset G.$$

Conjugation on H induces a map

$$N(\mathrm{Ker}) \to \mathrm{Aut}(\Gamma).$$

Definition 4.6. *Let $N_\Phi \subseteq N(\mathrm{Ker}) \subseteq G$ be the pullback of $\mathrm{Aut}_\Phi(\Gamma)$ and let $N_\Psi \subseteq N(\mathrm{Ker}) \subseteq G$ be the pullback of $\mathrm{Aut}_\Psi(\Gamma)$ under the above displayed homomorphism.*

Lemma 4.7. *We have: $N_\Phi = N_\Psi$.*

Proof. It suffices to show that $\mathrm{Out}_\Phi(\Gamma) = \mathrm{Out}_\Psi(\Gamma)$ for $\Gamma \in \mathfrak{G}$. We do this on a case by case basis.

1. $\Gamma = A_4$, $\mathrm{Out}(A_4) = \mathbf{Z}/2\mathbf{Z}$. There is a unique projective representation of A_4 of dimension 2 and a unique linear representation of dimension 3, and thus $\mathrm{Out}_\Phi(\Gamma) = \mathrm{Out}_\Psi(\Gamma) = \mathbf{Z}/2\mathbf{Z}$.
2. $\Gamma = S_4$, $\mathrm{Out}(S_4) = 1$. In this case one trivially has $\mathrm{Out}_\Phi(\Gamma) = \mathrm{Out}_\Psi(\Gamma) = 1$.
3. $\Gamma = A_5$, $\mathrm{Out}(A_5) = \mathbf{Z}/2\mathbf{Z}$. There are two inequivalent projective representations of A_5 of dimension 2 which are permuted by the outer automorphism group. Correspondingly, the outer automorphism group permutes the two 3-dimensional linear representations of A_5. Thus $\mathrm{Out}_\Phi(\Gamma) = \mathrm{Out}_\Psi(\Gamma) = 1$.
4. $\Gamma = D_n$, $\mathrm{Out}(D_n) = \mathrm{Out}(C_n)/\pm 1 = (\mathbf{Z}/n\mathbf{Z})^\times/\pm 1$. The outer automorphism group acts freely transitively on the set of 2-dimensional representations of D_n, and thus neither Φ nor X are preserved by any outer automorphisms. Hence $\mathrm{Out}_\Phi(\Gamma) = \mathrm{Out}_\Psi(\Gamma) = 1$.

□

In light of this lemma, we denote N_Φ (and correspondingly N_Ψ) by N.

Remark: In section 4.3, we study inductions of the (linear) representation X to G; in particular, the decomposition of $\mathrm{Ind}_H^G X$ into irreducibles with respect to $N = N_\Psi$. The utility of Lemma 4.7 is that from these calculations we are able to make conclusions about *projective* liftings of the representation V to $N = N_\Phi$, for example, Lemma 5.3.

We note the following consequence of Lemma 4.7.

Lemma 4.8. *The projective representation Φ' of H lifts to a projective representation of N. The linear representation Ψ' of H lifts to a linear representation of N.*

Proof. Since the corresponding morphisms $N \to \mathrm{Aut}_\Phi(\Gamma)$ and $N \to \mathrm{Aut}_\Psi(\Gamma)$ map H to $\mathrm{Inn}(\Gamma) = \Gamma$, it suffices to show that the representation Φ and Ψ considered as representations of Γ lift to $\mathrm{Aut}_\Phi(\Gamma)$ and $\mathrm{Aut}_\Psi(\Gamma)$, respectively. The claim for Φ follows from Lemma 4.2, and the claim for Ψ follows from Lemma 4.4, and the fact that $\mathrm{Out}_\Phi(\Gamma)$ has order one or two for all Γ and is thus cyclic. □

Remark: Note that the action on N factors through $\mathrm{Inn}(\Gamma) = \Gamma$ in all cases with the possible exception of $\Gamma = A_4$, when the action of N may factor through S_4 or A_4, depending whether the map $N(\mathrm{Ker}) \to \mathrm{Out}(\Gamma)$ is surjective or trivial.

Remark: It should be noted that the proof of Lemma 4.7 is not a formal consequence of any general relation between irreducible projective representations and the linear representations corresponding to their adjoints. In particular, $N_\Phi \neq N_\Psi$ for a general n-dimensional projective representation Φ and its corresponding linear adjoint representation Ψ of dimension n^2-1, even if the latter representation is irreducible. For example, the group $\Gamma = A_6$ admits four inequivalent faithful irreducible projective representations of dimension 3, which as a set are acted upon freely transitively by $\mathrm{Out}(A_6) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Thus, if PV is any 3-dimensional irreducible projective representation of A_6, then $\mathrm{Out}_\Phi(A_6) = 1$. On the other hand, A_6 has only two inequivalent irreducible 8-dimensional representations. Since one of them is the linear representation Ψ of A_6 on $\mathrm{Hom}'(PV, PV)$, there exists at least one outer automorphism of A_6 which preserves Ψ, and hence $\mathrm{Out}_\Psi(A_6) = \mathbf{Z}/2\mathbf{Z}$. In particular, there exists an 8-dimensional representation of $A_6.2$ lifting Ψ which does not arise from the adjoint of a three dimensional projective representation. By $A_6.2$ we mean the extension corresponding to an "exotic" automorphism of A_6 rather than S_6 (S_6 admits no eight dimensional irreducible representations).

4.3 Representations Induced from Normal Subgroups

Let $\Gamma \in \mathfrak{S}$, and let X be the representation space of definition 4.5 and, as in the previous section, we let Ψ denote the (E-linear, irreducible, faithful) representation of Γ on X. Let G be a finite group, and H a normal subgroup of G equipped with a surjective homomorphism to Γ. View the representation space X as a representation space for H via this surjective homomorphism, and — as in the previous section — denote by Ψ' the ensuing representation of H on X. For a given $g \in G$, let $X_g := gX$ denote the vector space whose vectors are written $\{gx \mid x \in X\}$ and such that the natural mapping $X \to gX$ ($x \mapsto gx$) is an isomorphism of E-vector spaces. We define a representation "Ψ^g" of H on X_g by composing the Ψ-action of H on X with the automorphism of H given by conjugation by g. Specifically, for $h \in H$ and gx an element of X_g the action is given by the rule: $h\star gx := gx'$ where $x' := (g^{-1}hg) \cdot x \in X$ for $x \in X$ and $h \in H$. In the special case when $g = 1$, the space X_g is identified with X.

Let $\mathcal{R} \subset G$ be a representative system for left cosets of H in G and let $g \in G$. We have an isomorphism of $E[G]$-modules:

$$\mathrm{Ind}_H^G X_g = E[G] \otimes_{E[H]} X_g \xrightarrow{\simeq} \bigoplus_{\gamma\in\mathcal{R}} \gamma X_g$$

where the action of G on the right-hand side is given by the evident formula. In particular, we have:

$$\mathrm{Ind}_H^G X \xrightarrow{\simeq} \bigoplus_{\gamma\in\mathcal{R}} \gamma X$$

where, again, the action of G is given by the evident formula; the action of H — in contrast — has the feature of preserving the direct sum decomposition on the right-hand side of the display above, the representation of H on the summand X_g being Ψ^g as described above.

Lemma 4.9. *Let $V \subseteq \mathrm{Ind}_H^G X$ be an irreducible sub-representation, and let g be any element of G. We have natural isomorphisms of E-vector spaces*

$$\mathrm{Hom}_H(V, X_g) \xrightarrow{\simeq} \mathrm{Hom}_G(V, \mathrm{Ind}_H^G X_g) \xrightarrow{\simeq} \mathrm{Hom}_G(V, \mathrm{Ind}_H^G X) \neq 0.$$

Proof. The first isomorphism comes from Frobenius reciprocity; the second is induced from the inverse to the natural E-vector space isomorphism $X \simeq X_g$ that intertwines the H representation Ψ on X to Ψ^g on X_g. ∎

Remark: Comparing the isomorphisms above for different elements $g, g' \in G$ gives us canonical isomorphisms of E-vector spaces

$$\mathrm{Hom}_H(V, X_g) \xrightarrow{\simeq} \mathrm{Hom}_G(V, \mathrm{Ind}_H^G X_g) \xrightarrow{\simeq} \mathrm{Hom}_H(V, X_{g'}) \xrightarrow{\simeq} \mathrm{Hom}_G(V, \mathrm{Ind}_H^G X_{g'}).$$

Recall the group N that fits into the sequence

$$\{1\} \subset \mathrm{Ker} \subset H \subset N \subset G$$

as discussed in the previous subsection.

Lemma 4.10. *1. Let $U \subseteq \mathrm{Ind}_H^N X$ be an irreducible representation of N. Then $\mathrm{Ind}_N^G U$ is an irreducible representation of G.*

2. Let $V \subseteq \mathrm{Ind}_H^G X$ be an irreducible representation of G. Then there is an irreducible representation $U \subseteq \mathrm{Ind}_H^N X$ of N and an isomorphism

$$V \xrightarrow{\simeq} \mathrm{Ind}_N^G U$$

of G-representations. (I.e., all irreducible representations in $\mathrm{Ind}_H^G X$ "come from" irreducible sub-representations of N acting on $\mathrm{Ind}_H^N X$.)

Proof. By definition, $\mathrm{Ind}_H^N X$ restricted to H is isomorphic to $\bigoplus_{g\in\mathcal{R}(N/H)} X_g \simeq X^{[N:H]}$ where $\mathcal{R}(N/H) \subset N$ is a representative system of H-cosets. Thus $\dim(U) = \dim(X) \cdot \dim \mathrm{Hom}_H(X, U)$. There is at least one irreducible representation V contained in $\mathrm{Ind}_N^G U$ such that $\mathrm{Res}_N V$ contains a copy of U. By Lemma 4.9 it follows that

$$\dim \mathrm{Hom}_H(V, X_g) = \dim \mathrm{Hom}_H(V, X) \geq \dim \mathrm{Hom}_H(U, X)$$

for all g, and thus as the number of distinct representations X_g is equal to G/N,

$$\dim(V) \geq [G : N] \dim(U) = \dim(\mathrm{Ind}_N^G U),$$

from which it follows that $V = \mathrm{Ind}_N^G U$ is irreducible. On the other hand, if $V \subseteq \mathrm{Ind}_H^G X$ is irreducible, then $U \subseteq V|_N$ for some irreducible N-representation U, and then $\mathrm{Hom}_G(V, \mathrm{Ind}_N^G U)$ is non-zero and $V = \mathrm{Ind}_N^G U$. ◻

4.4 Elements of order two

As in the previous subsection, let G be a finite group, $H \subset G$ a normal subgroup equipped with a homomorphism onto Γ with kernel denoted Ker and let $\mathrm{Ker} \subset H \subset N \subset G$ be as described previously; we also have the faithful irreducible projective representation Φ of Γ on V, and the corresponding linear representation Ψ of Γ on X. By an *involution* in a group we mean an element of order two.

Definition 4.11. *An involution $c \in N \subset G$ will be called* even *if c lies in the kernel of the map $N \longrightarrow \mathrm{Aut}(\Gamma)$.*

Let $\mathcal{C} \subset G$ be the conjugacy class of an involution in G; thus if $c \in \mathcal{C}$, $\mathcal{C} = \{gcg^{-1} \mid g \in G\}$.

Lemma 4.12. *Suppose that:*

1. *No $c \in \mathcal{C}$ is even.*
2. *There exists at least one $c \in \mathcal{C}$ such that $c \notin N$.*

Then if $V \subseteq \mathrm{Ind}_H^G X$ is an irreducible representation then:

1. *If $\Gamma = D_n$, $\dim(V|c = 1) = \dim(V|c = -1) = \frac{1}{2}\dim(V)$.*
2. *If $\Gamma \in \{A_4, S_4, A_5\}$, $\dim(V|c = -1) < \frac{2}{3}\dim(V)$.*

Proof. Note that $\dim(V|c=-1)$ does not depend on the choice of $c \in \mathcal{C}$, and so we assume that $c \notin N$. Since H is normal in G, H has index two in $\langle H, c\rangle$. Let U be an irreducible constituent of $\mathrm{Res}_{\langle H,c\rangle} V$. The representation $\mathrm{Res}_H U$ is a product of conjugates X_g of X for some collection of elements $g \in G$ (not necessarily distinct). By Frobenius reciprocity it follows that U injects into $\mathrm{Ind}_H^{\langle H,c\rangle} X_g$ for some g. If $\mathrm{Ind}_H^{\langle H,c\rangle} X_g$ is irreducible, then

$$\dim(U|c=1) = \dim(U|c=-1),$$

and we are done. If $\mathrm{Ind}_H^{\langle H,c\rangle} X_g$ is reducible, then it decomposes as a direct sum of two representations each of dimension $\dim(X) \in \{2,3\}$. The relations $\dim(U|c=-1) = \frac{1}{2}\dim(U)$ if $\dim(X)=2$ or $\dim(U|c=-1) \le \frac{2}{3}\dim(U)$ if $\dim(X)=3$ are then equivalent to the condition that c does not act as $+1$ or -1 on U. Now

$$U = \mathrm{Res}_H(U) \subset \mathrm{Res}_H \mathrm{Ind}_H^{\langle H,c\rangle} X_g = X_g \oplus cX_g = X_g \oplus X_{cg}.$$

Since U is clearly neither X_g nor cX_g (as they are not preserved by c) it follows that there must be an isomorphism of H representations $X_g \simeq X_{cg}$, or equivalently, an isomorphism of H representations $X \simeq X_{g^{-1}cg}$. It follows that $g^{-1}cg \in N$. If, in addition, c acts centrally on U, then the irreducible $\langle H,c\rangle$ subrepresentations of $X_g \oplus cX_g$ are given explicitly by the diagonal and anti-diagonal subspaces. The corresponding isomorphism $X \to X_{g^{-1}cg}$ is therefore the identity. It follows that the image of $g^{-1}cg$ in $\mathrm{Aut}(\Gamma)$ is trivial, and hence $g^{-1}cg$ is *even*, a contradiction.

To prove the *inequality* in the case when $\dim(X)=3$, it suffices to show that there exists at least one representation $U \subseteq \mathrm{Res}_{\langle H,c\rangle} V$ with $U = \mathrm{Ind}_H^{\langle H,c\rangle} X_g$, since then $\dim(U|c=-1) = \frac{1}{2}\dim(U) < \frac{2}{3}\dim(U)$. For this, note that there is at least one U such that $X \subseteq \mathrm{Res}_H U$. Yet if $\mathrm{Ind}_H^{\langle H,c\rangle} X$ is reducible, then by arguing as above we deduce that $c \in N$. By assumption $c \notin N$ and we are done. $\square$

5 Artin Representations

Fix a Galois extension $K/\mathbf{Q}$, an algebraically closed extension $E/\mathbf{Q}_p$, and an irreducible representation

$$\rho : G_K \to \mathrm{GL}_2(E)$$

with finite image. Let $\mathrm{Proj}(\rho)$ denote the associated two dimensional projective representation, and $\mathrm{ad}^0(\rho)$ the associated three dimensional linear representation. Let L denote the fixed field of the kernel of $\mathrm{Proj}(\rho)$, which is also the fixed field of the kernel of $\mathrm{ad}^0(\rho)$. By a well known classification, the image of $\mathrm{Proj}(\rho)$ belongs to the set $\mathfrak{S} := \{A_4, S_4, A_5, D_n, n \ge 3, \mathrm{odd}\}$. We make the following assumptions about the fields K and L.

1. p is totally split in K, and L/K is unramified at all primes above p.
2. If $v|p$, the eigenvalues α_v, β_v of $\rho(\mathrm{Frob}_v)$ are distinct (ρ is v-distinguished).

Fix once and for all an ordering of the pair α_v, β_v, and let $L_v = \alpha_v E$.

Definition 5.1. *Say that the representation ρ* descends to *$K^+ \subseteq K$ if and only if there exists a Galois representation $\varrho : G_{K^+} \to \mathrm{GL}_2(E)$ such that $\varrho|_{G_K} = \rho \otimes \chi$ where χ is a character of K. Similarly, say $\mathrm{Proj}(\rho)$ descends to K^+ if it is the restriction of some two dimensional projective representation of G_{K^+}.*

Remark: The image of ϱ can be strictly larger than the image of ρ, as can be seen by taking ϱ to be a $G_{\mathbf{Q}}$-representation with projective image S_4 and taking K to be the corresponding quadratic subfield.

The goal of this section is to give the proof of the main theorem:

Theorem 5.2. *Suppose that ρ admits infinitesimally classical deformations of minimal level. Assume the strong Leopoldt conjecture. Then one of the following holds:*

1. *There exists a character χ such that $\chi \otimes \rho$ descends to an odd representation over a totally real field.*
2. *The projective image of ρ is dihedral. The determinant character descends to a totally real field $H^+ \subseteq K$ with corresponding fixed field H such that*
   (*i*) *H/H^+ is a CM extension.*
   (*ii*) *At least one prime above p in H^+ splits in H.*
3. *The representation $\chi \otimes \rho$ descends to a field containing at least one real place at which $\chi \otimes \rho$ is even.*

5.1 Notation

This notation will be used for the remainder of the paper. Recall that L is the fixed field of the kernel of $\mathrm{Proj}(\rho)$. Let M be the Galois closure of L, and let $G = \mathrm{Gal}(M/\mathbf{Q})$. Let $H = \mathrm{Gal}(M/K)$. The group H is normal in G by construction. Let D be the decomposition group at some fixed prime v dividing p. The representation $\mathrm{Proj}(\rho)$ gives rise to a projective representation Φ of H, that factors through a group $\Gamma \in \mathfrak{S}$. Recall that $N = N_{\Phi}$ is the group defined as in Lemma 4.7. Let K^+ be the fixed field of N, so that

Lemma 5.3. $\mathrm{Proj}(\rho)$ *descends to K^+.*

5.2 Lifting projective representations

Since the deformation theory of ρ only depends on $\mathrm{Proj}(\rho)$, we turn our attention to projective representations. Recall the following theorem of Tate.

Theorem 5.4 (Tate). *Let K be a number field and E an algebraically closed field. Any continuous homomorphism $G_K \to \mathrm{PGL}_n(E)$ lifts to a continuous homomorphism $G_K \to \mathrm{GL}_n(E)$.*

Proof. Tate's result will follow if we show that $H^2(G_K, E^*)$ vanishes (here the action of G_K on E^* is trivial). Since E is algebraically closed, the multiplicative group E^* is isomorphic to an extension of a uniquely divisible group by a group isomorphic to $\mathbf{Q}/\mathbf{Z}$, so $H^2(G_K, E^*) = H^2(G_K, \mathbf{Q}/\mathbf{Z}) = H^3(G_K, \mathbf{Z})$. But the latter group vanishes. For this, see page 77, Cor. 4.17 in [29], where it is shown that $H^r(G_K, \mathbf{Z})$ vanishes for all number fields K and all odd values of r. ∎

Lemma 5.5. *The representation $\chi \otimes \rho$ descends to K^+ for some character χ.*

Proof. By Lemma 5.3, the representation $\mathrm{Proj}(\rho)$ can be descended to a representation:

$$\mathrm{Proj}(\rho)^+ : G_{K^+} \to \mathrm{PGL}_2(E),$$

then by Theorem 5.4, the representation $\mathrm{Proj}(\rho)^+$ lifts to a linear representation ϱ^+ of G_{K^+}. Since any two lifts of a projective representation are equivalent up to a character, it follows that $\varrho := \varrho^+|_{G_K} = \rho \otimes \chi$, and the result follows. ∎

5.3 Complex conjugation in G

Let $\mathcal{C}$ be the conjugacy class of complex conjugation elements of G.

Proposition 5.6. *One of the following conditions hold:*

1. *Every $c \in \mathcal{C}$ is not even,*
2. *There exists at least one $c \in \mathcal{C}$ such that $c \notin N$,*
3. *The projective representation* $\mathrm{Proj}(\rho)$ *descends to a representation over a field containing at least one real place which is even,*
4. *The projective representation* $\mathrm{Proj}(\rho)$ *descends to an odd representation over a totally real field.*

Proof. Suppose that every $c \in \mathcal{C}$ lies in N. Then the fixed field K^+ of N is totally real, and $\mathrm{Proj}(\rho)$ descends to K^+ (and is either odd or even at each place). If c is even, then consider the descended representation $\varrho = \chi \otimes \rho$ at the corresponding real place. Since c acts trivially by conjugation on Γ, it follows that c acts centrally on this representation, and thus via ± 1 (by Schur's lemma). Hence, possibly after a twist, ϱ is even at this place. $\square$

5.4 Selmer Groups of Artin Representations

Recall (from section 3) the following notation:

- the $E[G]$-module formed from *global units* of M: $U_{\mathrm{Global}} := \mathcal{O}_M^* \otimes_{\mathbf{Z}} E$;
- the $E[D_v]$-module formed from *local units* of M_v for places v dividing p: $U_v := \widehat{\mathcal{O}}_{M_v}^* \otimes_{\mathbf{Z}_p} E$;
- and the natural homomorphism

$$\iota = \iota_p = \prod_{v|p} \iota_v : U_{\mathrm{Global}} \longrightarrow \prod_{v|p} U_v = \oplus_{v|p} U_v = U_{\mathrm{Local}},$$

  which the classical Leopoldt conjecture asserts is injective.

Assume the classical Leopoldt conjecture, and form the exact sequence

$$0 \to U_{\mathrm{Global}} \xrightarrow{\iota_p} U_{\mathrm{Local}} \to \Gamma_M \to 0.$$

The E-vector space Γ_M can be identified, via Class Field Theory, with the tensor product of the Galois group of the maximal $\mathbf{Z}_p$-power extension of M over $\mathbf{Z}_p$ with E.

Let $F \subset E$ be fields as in subsection 3.1, and let $\Lambda_F := F[G]$ and $\Lambda := \Lambda_F \otimes_F E$. Make the identification of left-Λ-modules $\Lambda \simeq U_{\mathrm{Local}}$ (compatible with preferred F-structures; see subsection 3.1). Consequently, given the classical Leopoldt conjecture, U_{Global} will be identified with some left ideal $I_{\mathrm{Global}} \subset \Lambda$, and Γ_M above will be identified with the quotient $\Lambda / I_{\mathrm{Global}}$. After these identifications, and setting $I := I_{\mathrm{Global}}$, the exact sequence displayed above may be (cryptically) written:

$$0 \to I \ \to \ \Lambda \to \Lambda / I \to 0.$$

Recall (from subsection 2.1 that $W := \mathrm{Hom}'(V, V) \subset \mathrm{Hom}(V, V)$ denotes the G_K-stable hyperplane in $\mathrm{Hom}(V, V)$ consisting of endomorphisms of V of trace zero. By the inflation-restriction sequence there is an isomorphism

$$H^1(K, W) \simeq H^1(M, W)^H = H^1(M, \mathrm{Hom}'(V, V))^H.$$

Let $H^1_{\Sigma,(p)}(K,W)$ denote the Selmer group without any local conditions at p. Since we may consider minimal deformations $\Sigma = \emptyset$ (see Corollary 2.9, and the remarks following Definition 2.14), the group $H^1_{\Sigma,(p)}(K,W)$ is therefore isomorphic to

$$\mathrm{Hom}_E(\Gamma_M, \mathrm{Hom}'(V,V))^H \simeq \mathrm{Hom}_E(\Lambda/I, \mathrm{Hom}'(V,V))^H,$$

the right-hand displayed module coming from the identification made above. The away from p Selmer group $H^1_{\Sigma,(p)}(K,W)$ fits into an exact sequence

$$0 \to H^1_{\Sigma,(p)}(K,W) \to \mathrm{Hom}_E(U_{\mathrm{Local}}, \mathrm{Hom}'(V,V))^H \to \mathrm{Hom}_E(U_{\mathrm{Global}}, \mathrm{Hom}'(V,V))^H,$$

which — after our identifications — can be written

$$0 \to \mathrm{Hom}_E(\Lambda/I, \mathrm{Hom}'(V,V))^H \to \mathrm{Hom}_E(\Lambda, \mathrm{Hom}'(V,V))^H \to \mathrm{Hom}_E(I, \mathrm{Hom}'(V,V))^H.$$

We have a canonical identification

$$\mathrm{Hom}_E(U_{\mathrm{Local}}, \mathrm{Hom}'(V,V)) = \mathrm{Hom}_E(\oplus_{v|p} U_v, \mathrm{Hom}'(V,V)) = \prod_{v|p} \mathrm{Hom}_E(U_v, \mathrm{Hom}'(V,V)).$$

For each $v|p$ we have the natural homomorphism $\mathrm{Hom}'(V,V) \to \mathrm{Hom}(L_v, V/L_v)$.

Recall that $W_v^0 := \mathrm{Ker}(\mathrm{Hom}'(V,V) \to \mathrm{Hom}(L_v, V/L_v))$, and that we will be using these mappings to cut out the ordinary piece of the Selmer modules. Recall, as well, the local Selmer conditions $\mathcal{L}_v \subseteq H^1(D_v, W)$ defined in subsection 2.1,

Lemma 5.7. *Let $\gamma \in H^1_{\Sigma,(p)}(K,W)$. Then $\gamma \in \mathcal{L}_v$ for $v|p$ if and only if the corresponding morphism $\phi \in \mathrm{Hom}_H(\Lambda, \mathrm{Hom}'(V,V))$ lies in the kernel of the composite map*

$$\begin{tikzcd}
\mathrm{Hom}_H(\Lambda, W) \simeq \mathrm{Hom}_H(U_{\mathrm{Local}}, W) \arrow[r] \arrow[dr] & \mathrm{Hom}(U_v, W) \arrow[d] \\
 & \mathrm{Hom}(U_v, W/W_v^0)
\end{tikzcd}$$

Denote the intersection in $\mathrm{Hom}_H(\Lambda, W)$ of these kernels for all v by $\mathrm{Hom}^0_H(\Lambda, W)$.

Proof. This is simply a matter of tracing the definition of $\mathcal{L}_v$. $\square$

Lemma 5.8. *The group $\mathrm{Hom}^0_H(\Lambda, W)$ is the subset of homomorphisms ϕ such that $\phi([\sigma]) \in W^0_{\sigma(v)}$ for all $\sigma \in G$.*

By Frobenius reciprocity we may re-write the Selmer group $H^1_{\Sigma,(p)}(K,W)$ as

$$\mathrm{Hom}_G(\Lambda/I, \mathrm{Ind}_H^G W).$$

Recall that we have a chosen place v of M above p, and we have also chosen a representative system $\mathcal{R}(G/H) \subset G$ of left H-cosets in G.

Lemma 5.9. *After the identification we have made, we have that*

$$H^1_\Sigma(K,W) = \mathrm{Hom}_G(\Lambda/I, \mathrm{Ind}_H^G W) \bigcap \mathrm{Hom}^0_G(\Lambda, \mathrm{Ind}_H^G W),$$

where the adornment 0 *in the second homomorphism group indicates morphisms* $\phi : \Lambda \to \mathrm{Ind}_H^G W$ *such that*

$$\phi([1]) \in \bigoplus_{\sigma \in \mathcal{R}(G/H)} W^0_{\sigma(v)}$$

Definition 5.10. *Let* $Y = \mathrm{Ind}_H^G W$ *and* $Y^0 = \bigoplus_{\sigma\in\mathcal{R}(G/H)} W^0_\sigma$.

Corollary 5.11. *In the notation of section A.4, we have a canonical isomorphism*

$$H^1_\Sigma(K,W) \simeq \mathrm{Hom}_G(\Lambda/I, Y; Y^0).$$

If $\gamma \in H^1_\Sigma(K,W)$, then the infinitesimal Hodge-Tate weights are given as follows. Let ϕ denote the corresponding element of $\mathrm{Hom}_G(\Lambda, Y)$, so $\phi([1]) \in Y^0$. Recall there is a canonical homomorphism $W^0_v \xrightarrow{\epsilon} \mathrm{End}_E(L_v) = E$. The composite map

$$\pi : H^1_{\Sigma,(p)}(K,W) \to Y^0 \to \bigoplus_{\sigma\in\mathcal{R}(G/H)} E,$$

is exactly the map of Definition 2.13.

Definition 5.12. *Let* Y^{00} *denote the kernel of the map* $Y^0 \to \bigoplus_{\sigma\in\mathcal{R}(G/H)} E$.

The spaces Y^0 and Y^{00} are, by construction, left rational D-submodules of Y. To orient the reader, choosing an appropriate local basis everywhere, the spaces Y^0 and Y^{00} can be thought of as $|G/H|$ copies of the upper triangular trace zero matrices and upper nilpotent matrices respectively. The Bloch–Kato Selmer module (usually denoted $H^1_{\Sigma,f}(K,W)$) is identified with $\mathrm{Hom}_G(\Lambda/I, Y; Y^{00})$.

5.5 Involutions respecting irreducible subspaces

Now that we have identified the various Selmer groups of interest to us as vector spaces of the form $\mathrm{Hom}_G(\Lambda/I, Y; Y^0)$ and $\mathrm{Hom}_G(\Lambda/I, Y; Y^{00})$, our next goal is to prove our main theorem by making use of the Strong Leopoldt Conjecture which allows us to compare the dimensions of these spaces with the dimensions of spaces of homomorphisms $\mathrm{Hom}_G(\Lambda/J, Y; Z)$, where J is a left $E[G]$-ideal such that we have an isomorphism of left $E[G]$-modules $J \simeq I$ and Z is an appropriate left rational D-submodule (test object) related to Y^0 and Y^{00}. (Note that the appropriate test objects are left modules, as the genericity hypothesis will be applied in the dual context of Lemma A.14.) To do this recall by Lemma A.15 we need to construct certain subspaces Y'_i for each V_i-isotypic component Y_i of Y. Since the irreducible representations $V \subseteq Y$ are not apparent from the representation of Y as $\mathrm{Ind}_H^G W$, we construct the appropriate spaces Y'_i by choosing involutions on certain subspaces of Y which we show stabilize each irreducible representation $V \subseteq Y$. Recall that $X \subseteq W$ is the unique irreducible representation in W on which Γ acts faithfully; it equals W unless $\Gamma = D_n$ when X has dimension two. We may write

$$\mathrm{Ind}_H^G X = \mathrm{Ind}_N^G \mathrm{Ind}_H^N X = \bigoplus_{g\in\mathcal{R}(G/N)} g\mathrm{Ind}_H^N X$$

where $\mathcal{R}(G/N) \subset G$ is a representative system for the cosets of N in G. Since N/H is not necessarily normal in G/H, the vector space $g\mathrm{Ind}_H^N X$ is not a representation of N, but rather of gNg^{-1}. Explicitly we have an isomorphism

$$g\mathrm{Ind}_H^N X \simeq \mathrm{Ind}_H^{gNg^{-1}} X_g.$$

Recall by Lemma 4.10 that if $V \subseteq \mathrm{Ind}_H^G X$ is irreducible, then $V = \mathrm{Ind}_N^G U$ for some irreducible $U \subseteq \mathrm{Ind}_H^N X$. Thus

$$V = \mathrm{Ind}_H^G U = \bigoplus_{g \in \mathcal{R}(G/N)} U_g, \qquad U_g \subseteq \mathrm{Ind}_H^{gNg^{-1}} X_g.$$

Here U_g is an irreducible gNg^{-1} submodule of $\mathrm{Ind}_H^{gNg^{-1}} X_g$. From this we deduce the following:

Lemma 5.13. *To construct an involution ι on $\mathrm{Ind}_H^G X$ preserving the irreducible G representations it suffices to construct a collection of involutions ι_g on $\mathrm{Ind}_H^{gNg^{-1}} X_g$ for each $g \in \mathcal{R}(G/N)$ which preserve the irreducible gNg^{-1} representations inside $\mathrm{Ind}_H^{gNg^{-1}} X_g$.*

Proof. Let $\iota = \bigoplus \iota_g$. Then ι_g preserves each U_g because by definition U_g is a irreducible gNg^{-1} submodule of $\mathrm{Ind}_H^{gNg^{-1}} X_g$. Thus the map ι preserves their direct sum, which is $\mathrm{Ind}_N^G U$. By Lemma 4.10 the module $\mathrm{Ind}_N^G U$ is irreducible, and moreover all irreducibles are of this form. □

5.6 Construction of involutions respecting irreducible subspaces

Let $\mathcal{R}(N/H)$ be a representative system of left cosets of H in N. By definition of the group N there is an isomorphism

$$\mathrm{Ind}_H^{gNg^{-1}} X_g = \bigoplus_{n \in \mathcal{R}(N/H)} g \cdot n \cdot X, \qquad \mathrm{Res}_H \mathrm{Ind}_H^{gNg^{-1}} X_g \simeq (gX)^{[N:H]}.$$

To construct ι_g we begin by constructing an involution on gX for any g. We do this by more generally constructing an involution on gW for any g that preserves the decomposition $gW = gX \oplus g\epsilon$ when $\Gamma = D_n$.

Let v be the place above p "corresponding to" g in $\mathcal{R}(G/N)$; i.e., if $D = \langle \phi \rangle$ is the decomposition group corresponding to our fixed choice of place above p then the decomposition group D_v is isomorphic to $gDg^{-1} = \langle g\phi g^{-1} \rangle$. Let $L_v \subseteq W_v$ denote the special D_v-invariant line. By construction of induced representations, the action of H on $W_v \subseteq Y$ is conjugated by g, and thus $L_v \subseteq Y$ is ϕ-invariant. Thus for the rest of this subsection we drop the subscript v from L_v and from the sequence of subspaces $W_v^{00} \subset W_v^0 \subset W_v$. Recall that these spaces fit into the following exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & W^{00} \arrow[r, equal] \arrow[d, hook] & \mathrm{Hom}(V/L, L) \arrow[d, hook] & & \\
0 \arrow[r] & W^0 \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \mathrm{Hom}(L, V/L) \arrow[r] \arrow[d, equal] & 0. \\
0 \arrow[r] & \mathrm{Hom}(L, L) \arrow[r] & \mathrm{Hom}(L, V) \arrow[r] & \mathrm{Hom}(L, V/L) \arrow[r] & 0
\end{tikzcd}$$

There is a canonical identification $\mathrm{Hom}(L, L) = E$.

Lemma 5.14. *Let ψ be any non-trivial element of order two in Γ such that the commutator $[\psi, \phi] \neq 0$. Then*

$$W^{\psi=-1} \cap W^{00} = 0.$$

Moreover, $\dim(W^{\psi=-1}) = 2$. *If $\Gamma = D_n$, then*

1. $\dim(X^{\psi=-1}) = 1$, $\dim(\epsilon^{\psi=-1}) = 1$.
2. $\dim(W^{\psi=-1} \cap W^0) = 1$, *and the projection map* $W^{\psi=-1} \cap W^0 \to \epsilon$ *is an isomorphism.*
3. *Suppose that ϕ has order two. In the diagram:*

$$\begin{tikzcd}
& W^{\psi=-1} \cap W^0 \arrow[dl] \arrow[dr] & \\
\mathrm{Hom}(L, L) = E \arrow[rr, dotted, "\eta_\psi"] & & \epsilon
\end{tikzcd}$$

the down arrows induced from the natural surjections $W^0 \to \mathrm{Hom}(L, L)$ and $W \to \epsilon$ are isomorphisms, and there is a corresponding non-zero map $\eta_\psi : E \to \epsilon$ making the diagram commute. If ψ, ξ are two elements of Γ of order two distinct from ϕ and from each other, then $\eta_\psi \neq \eta_\xi$.

Proof. Let ψ be any element of order two. If $W^{\psi=-1} \cap W^{00} \neq 0$, then in particular ψ preserves $W^{00} = \mathrm{Hom}(V/L, L)$. Yet $\psi W^{00} = \mathrm{Hom}(V/\psi L, \psi L)$, which can equal W^{00} if and only if $\psi L = L$. Yet $\phi L = L$, and thus $[\psi, \phi]$ acts trivially on L. In particular (lifting ϕ, ψ to $\mathrm{GL}(V)$), $[\psi, \phi]$ is an element of finite order of determinant one with a fixed line L, which is therefore unipotent and hence trivial since $[\psi, \phi]$ has finite order.

Now let us assume that $\Gamma = D_n$. If ψ is any element of order two, then $\epsilon^{\psi=-1} = \epsilon$ has dimension one, and thus $X^{\psi=-1}$ also has dimension one. The identity $\dim(W^{\psi=-1} \cap W^0) = 1$ is automatic by the first part of the lemma and for dimension reasons. If ϕ has odd order, then $\epsilon = W^{\phi=1} \subseteq W^0$ and $\epsilon \subseteq W^{\psi=-1}$, and so $W^{\psi=-1} \cap W^0 = \epsilon$, and in particular the projection onto ϵ is non-trivial. Thus, from now on, we assume that ϕ has order two.

Sublemma 3. *The maps $W^0 \to \mathrm{Hom}(L,L) = E$ and $W^0 \to \epsilon$ are D-module homomorphisms with distinct kernels, and together identify W^0 with $E \oplus \epsilon$.*

Proof. By construction the maps are D-module homomorphisms. Since E is a $\phi = 1$ eigenspace and ϵ is a $\phi = -1$ eigenspace, the result is immediate. $\square$

Consider the composite maps $W^0 \cap W^{\psi=-1} \to \mathrm{Hom}(L,L) = E$ and $W^0 \cap W^{\psi=-1} \to \epsilon$.

Sublemma 4. *These maps are isomorphisms of vector spaces.*

Proof. The kernel of the first map is contained inside the kernel of the map $W^0 \to \mathrm{Hom}(L,L)$, which is W^{00}. Since $W^{00} \cap W^{\psi=-1} = 0$, the first map is injective, and hence an isomorphism by dimension counting.

The projection $W^0 \to \epsilon$ is a map of D-modules, and by Sublemma 3 the kernel is D-stable (equivalently: ϕ-stable). Thus the kernel of $W^0 \cap W^{\psi=-1} \to \epsilon$ is both ϕ and ψ stable. If it is non-zero, it must be a line which lands completely within X, since X is the kernel of the projection $W \to \epsilon$. Yet $[\phi, \psi]$ would fix this line and at the same time be a determinant one matrix with finite order, a contradiction. Thus the second map is injective, and hence an isomorphism. $\square$

This lemma concludes part (2), since we have shown that $W^0 \cap W^{\psi=-1} \to \epsilon$ is surjective. It remains to prove part (3). The existence of η_ψ is guaranteed by Sublemma 4. We note that the identification $W^0 = E \oplus \epsilon$ implies that $W^0 \cap W^{\psi=-1} \subset W^0$ is exactly the *graph* of the homomorphism η_ψ. Now two homomorphisms η_ψ and η_ξ are equal if and only if their graphs are the same. Suppose that

$$W^0 \cap W^{\psi=-1} = W^0 \cap W^{\xi=-1}.$$

If the projection of these spaces to X were zero, then $W^0 \cap W^{\psi=-1}$ would lie in ϵ, and then (since $W^0 = E \oplus \epsilon$) their projection to E would be zero, a contradiction. Thus the projection to X defines a line that is preserved by ψ and by ξ. Once more we arrive at a contradiction unless ψ and ξ commute. Yet any pair of elements in D_n (n odd) of order two do not commute, and thus we are done. $\square$

Thus, for each $g \in G$, we have constructed an involution on W_g by some element $\psi \in \Gamma$. For a fixed element g make a choice of ψ, and then lift ψ to an element $h \in H$. By definition, if $n \in N$ then $g \cdot n \cdot W \simeq W_g$. Thus the action of h and ϕ on $g \cdot n \cdot W$ is the same as that on fgW, and in particular, the conclusions of Lemma 5.14 apply to all the constituents of $\mathrm{Res}_H \mathrm{Ind}_H^{gNg^{-1}} W_g$.

Definition 5.15. *Define the involution ι_g on $\mathrm{Ind}_H^{gNg^{-1}} W_g$ to be map obtained from multiplication by h.*

By Lemma 5.13 we have constructed at least one involution ι on $\mathrm{Ind}_H^G X$ by compiling the ι_g's for $g \in \mathcal{R}(G/N)$.

If $\Gamma = D_n$, note that Γ_n always has at least three distinct non-commuting elements of order two, and they all act as -1 on ϵ, and so extend to involutions ι_g on $\mathrm{Ind}_H^G W$ preserving G-irreducible representations.

Fix such an involution $\iota : \mathrm{Ind}_H^G X \to \mathrm{Ind}_H^G X$.

Definition 5.16. *Let* $Y' := Y^{\iota=-1} = (\mathrm{Ind}_H^G W)^{\iota=-1}$.

Lemma 5.17. *Let* $V, V' \subset Y = \mathrm{Ind}_H^G W$ *be two irreducible* $E[G]$*-submodules. Then any isomorphism of* $E[G]$*-modules* $V \to V'$ *induces an isomorphism of vector spaces* $Y' \cap V \simeq Y' \cap V'$*. Moreover,* Y' *is generated by elements lying in the intersections* $Y' \cap V$ *for all irreducible* $E[G]$*-submodules* $V \subset Y$.

Proof. If $\Gamma = D_n$ and $V \subseteq \mathrm{Ind}_H^G \epsilon$, the result is trivial because then $Y' \cap V = V$ and $Y' \cap V' = V'$. Any irreducible $V \subseteq \mathrm{Ind}_H^G X$ is of the form $\mathrm{Ind}_N^G U$ for some irreducible N-representation $U \subseteq \mathrm{Ind}_H^N X$. Thus any isomorphism of $E[G]$-modules $V \to V'$ arises from an isomorphism of $E[N]$-modules $U \to U'$. Yet $Y' \cap U = U^{\iota=-1}$ is canonically defined in terms of the abstract N-structure of U and thus is preserved under isomorphism. Finally, since ι preserves each irreducible representation V inside Y (by Lemma 5.13, and the subsequent construction), it follows that any element of $Y' = Y^{\iota=-1}$ can be decomposed as a sum of $\iota = -1$ eigenvectors of irreducible representations $V \subset Y$, and hence into elements lying in the subspaces $Y' \cap V$. □

Note that as an H representation, W is either irreducible or the sum of two distinct irreducibles, and hence cyclic. Thus $Y = \mathrm{Ind}_H^G W$ is cyclic as a G-module, and hence admits an injection $Y \to \Lambda$ compatible with rational structures. It follows from Lemma 5.17 and Lemma A.15 that Y' is of the form $\mathrm{Hom}_G(\Lambda/J, Y)$ for some ideal $J \subset \Lambda$. Thus, in the language of section A.4, we may write

Corollary 5.18. *There is a decomposition* $Y' = \bigoplus S_i \otimes T_i^* \subset Y = \bigoplus V_i \otimes T_i^* \subset \Lambda$*, where, for each* i*,* $S_i \subset V_i$ *is an* E*–subspace, and* $T_i^* \subseteq V_i^*$ *is a rational* E*–subspace.*

5.7 The dimensions of the subspaces S_i

Lemma 5.19. *Let* $U_g \subseteq \mathrm{Ind}_H^{gNg^{-1}} X_g$ *be an irreducible* gNg^{-1}*-sub-representation. Then*

1. *If* $\Gamma \in \{A_4, S_4, A_5\}$*, then*
$$\dim(U_g^{\iota=-1}) = \frac{2}{3} \cdot \dim(U_g).$$
2. *If* $\Gamma = D_n$*, then*
$$\dim(U_g^{\iota=-1}) = \frac{1}{2} \cdot \dim(U_g).$$

If $V \subseteq \mathrm{Ind}_H^G X$ *is an irreducible* G*-representation, then*

1. *If* $\Gamma \in \{A_4, S_4, A_5\}$*, then*
$$\dim(V^{\iota=-1}) = \frac{2}{3} \cdot \dim(V).$$
2. *If* $\Gamma = D_n$*, then*
$$\dim(V^{\iota=-1}) = \frac{1}{2} \cdot \dim(V).$$

Finally, $Y' \cap Y^{00} = 0$*, where* Y^{00} *is as in Definition 5.12.*

Proof. The representation $U_g|H$ is equal to a number of copies of X_g, upon which ι_g is acting by construction by our chosen element of order two ψ. Thus the result follows directly from Lemma 5.14, in particular the claim that $\dim(W^{\psi=-1}) = 2$ and $\dim(X^{\psi=-1}) = 1$ when $\Gamma = D_n$. For V the result follows in the same way by restricting to each U_g. Note that for two distinct $g, g' \in G/H$ the modules gW, $g'W$ may be isomorphic but the ordering of the eigenvalues α, β may be reversed, (i.e., L_v may depend on $g \in G/H$ rather than $g \in G/N$). However, the proof of Lemma 5.14 only uses the fact that $\phi L_v = L_v$, and so is symmetric with respect to the two possible choices of special lines. Finally, the fact that $Y' \cap Y^{00} = 0$ also follows from Lemma 5.14, in particular the claim that $W^{\psi=-1} \cap W^{00} = 0$. $\square$

In terms of the decomposition of Y' in Corollary 5.18, we may read off information concerning the dimension of the spaces S_i from Lemma 5.19.

Corollary 5.20. *In the decomposition*

$$Y' = \bigoplus S_i \otimes T_i^* \subseteq \ Y = \bigoplus V_i \otimes T_i^* \subseteq \Lambda$$

of Corollary 5.18 (with rational E-subspaces $T_i^* \subseteq V_i^*$) we have that $S_i \subseteq V_i$ is a subspace such that:

1. *If $\Gamma \in \{A_4, S_4, A_5\}$, then $\dim(S_i) = \frac{2}{3}\dim(V_i)$.*
2. *If $\Gamma = D_n$ and $V_i \subseteq \mathrm{Ind}_H^G X$, then $\dim(S_i) = \frac{1}{2}\dim(V_i)$.*
3. *If $\Gamma = D_n$ and $V_i \subseteq \mathrm{Ind}_H^G \epsilon$, then $\dim(S_i) = \dim(V_i)$.*

6 Infinitesimally Classical Deformations of Artin Representations

Recall that we are working with a Galois extension $K/\mathbf{Q}$, an algebraically closed extension $E/\mathbf{Q}_p$, and an irreducible representation

$$\rho : G_K \to \mathrm{GL}_2(E)$$

with finite image. Recall that ρ admits an infinitesimally classical deformation if and only if there is a rational vector λ in $E^{[K:\mathbf{Q}]} = Y^0/Y^{00}$ that lies in the image of a non-zero element in $H^1_\Sigma(K, \mathrm{ad}^0(\rho)) = H^1_\Sigma(K, W)$ under π.

The subspace $Y^{00} \subset Y$ is rational, and by construction is a left D-submodule of Y. Furthermore, the same is true, by construction, of $Y^{00}(\lambda) := \pi^{-1}(\lambda E)$ (which is indeed a D-module since the action of D on Y^0/Y^{00} is trivial). If ρ admits such an infinitesimally classical deformation λ, we get the following inequality, which we record for future purposes as a lemma:

Lemma 6.1. *If ρ admits an infinitesimally classical deformation corresponding to λ, then*

$$\mathrm{Hom}_G(\Lambda/I, Y; Y^{00}(\lambda)) \neq 0$$

where $Y^{00}(\lambda)$ is a rational D-submodule of Y.

We shall prove the main theorem (Theorem 5.2) by contradiction. Specifically, we will assume the following

Hypothesis 1. *The Artin representation ρ admits an infinitesimally classical deformation λ and yet there is no character χ such that $\chi \otimes \rho$ descends to either to an odd representation over a totally real field or to a field containing at least one real place at which $\chi \otimes \rho$ is even. If the projective image of ρ is dihedral, assume further that the determinant character does not descends to a totally real field $H^+ \subseteq K$ with corresponding fixed field H such that*

(i) H/H^+ is a CM extension.

(ii) At least one prime above p in H^+ splits in H.

and end with a contradiction to this hypothesis.

6.1 Construction of the subspace $Y'(\lambda) \subset Y'$

We shall set about constructing a space $Y'(\lambda)$ by removing selected vectors from $Y' = Y^{\iota=-1}$. If $\lambda = 0$ then we let $Y'(\lambda) = Y'$, and all the proofs below continue to go through (alternatively, one may note that in this case Lemma 6.1 is true with *any* choice of λ). Henceforth we assume that $\lambda \neq 0$. Since $Y' \cap Y^{00} = 0$, the intersection $Y' \cap Y^{00}(\lambda)$ is at most one dimensional. Since $\dim(Y^{\iota=-1}) + \dim(Y^{00}) = \dim(Y)$, this dimension is exactly one. Let v_λ denote a a non-trivial vector in this intersection. The vector v_λ projects to a non-trivial isotypic component corresponding to some representation V_i, which we fix once and for all, and denote by V_λ. If $\Gamma = D_n$, we make the additional hypothesis that $V_\lambda \subseteq \mathrm{Ind}_H^G \epsilon$. That this is possible is the content of Lemma 5.14 part two, since the projection of v_λ to W_v^0 must land outside W_v^{00} for at least one v, and the map $W_v^0 \cap W_v^{\psi=-1} \to \epsilon$ is an isomorphism. Let π_{V_λ} denote a projection map $Y \to V_\lambda$ such that $\pi_{V_\lambda}(v_\lambda) \neq 0$.

Definition 6.2. *We define $S_i(\lambda) \subset V_i$ and thus $Y'(\lambda)$ as follows:*

1. *If $V_i \neq V_\lambda$, then $S_i(\lambda) := S_i$.*
2. *If $V_i = V_\lambda$, then $S_i(\lambda) :=$ a hyperplane in S_i that does not contain $\pi_{V_\lambda}(v_\lambda)$.*

Definition 6.3.

$$Y'(\lambda) = \bigoplus S_i(\lambda) \otimes T_i^* \subseteq \ Y' = \bigoplus S_i \otimes T_i^* \subseteq \Lambda$$

The earlier discussion gave us that $Y' \cap Y^{00}(\lambda)$ is one-dimensional and generated by v_λ and the construction of $Y'(\lambda)$ above guarantees that $v_\lambda \notin Y'(\lambda)$; thus $Y'(\lambda) \cap Y^{00}(\lambda) = 0$.

Proposition 6.4. *Assume Hypothesis 1. Then* $\dim(S_i(\lambda)) \geq \dim(V_i | c = -1)$.

Proposition 6.5. *The Strong Leopoldt Conjecture is in contradiction with Hypothesis 1.*

Theorem 5.2 follows from Propositions 6.4 and 6.5. We prove Propositions 6.4 and 6.5 in sections 6.2 and 6.4 respectively.

6.2 Proof of Proposition 6.4

By Hypothesis 1, ρ does not descend as in the main theorem, and hence if $\mathcal{C}$ is the conjugacy class of complex conjugation in G, then every $c \in \mathcal{C}$ is not even, and there exists at least one $c \in \mathcal{C}$ such that $c \notin N$.

In particular, the results of Lemma 4.12 apply.

Proposition 6.6. *Suppose that $\Gamma \in \{A_4, S_4, A_5\}$, then*

$$\dim(S_i(\lambda)) \geq \dim(S_i) - 1 = \frac{2}{3}\dim(V_i) - 1 \geq \dim(V_i|c = -1).$$

Proof. The last inequality follows from Lemma 4.12. ☐

Now let us consider the dihedral case. Suppose that $\Gamma = D_n$. If $V_i \subseteq \mathrm{Ind}_H^G X$, then $V_i \neq V_\lambda$, and thus

$$\dim(S_i(\lambda)) = \dim(S_i) = \frac{1}{2}\dim(V_i) = \dim(V_i|c = -1),$$

after applying Lemma 4.12. Thus it suffices to consider the case when $V_i \subseteq \mathrm{Ind}_H^G \epsilon$.

Remarks:

1. If $V_i \subseteq \mathrm{Ind}_H^G \epsilon$ and $V_i \neq V_\lambda$, then $\dim(S_i(\lambda)) = \dim(S_i) = \dim(V_i) \geq \dim(V_i|c = -1)$, which suffices,
2. If $V_i = V_\lambda$, then $\dim(S_i(\lambda)) = \dim(S_i) - 1 = \dim(V_\lambda) - 1$, which is $\geq \dim(V_\lambda|c = -1)$ if and only if c does not act centrally as -1 on V_λ.

From these calculations we have proven Proposition 6.4, except in the case where $\Gamma = D_n$ and V_λ has the property that c acts centrally as -1 on V_λ. The next lemma implies that under certain conditions we may choose V_λ so that c does not act centrally, completing the proof of the main theorem in these cases. For the remainder of the section we show that all other situations lead to case two of the main theorem, which completes the proof.

We postpone the proof of the following lemma to the next section.

Lemma 6.7. *Suppose that the action of ϕ on W_g is through an element of order two for all g, and that G admits at least one complex conjugation $c \notin N$. Then there exists a choice of involution ι such that the projection $\pi(\mathrm{Ind}_H^G \epsilon, v_\lambda)$ of v_λ to $\mathrm{Ind}_H^G \epsilon$ is not a -1 eigenvector for all conjugates c.*

Since we are assuming there exists a $c \in G$ such that $c \notin N$, if ϕ acts through an element of order two for all $g \in G/N$ we may choose a V_λ on which some conjugate of c does not act as -1. Yet then $\dim(V_\lambda|c = -1) \leq \dim(V_\lambda) - 1$ and we are done. Hence it remains in the final cases to establish a suitable descent.

Lemma 6.8. *Suppose that there exists at least one g such that ϕ acts on W_g through an element of odd order. Then ϵ descends to a quadratic character of a totally real subfield H^+ of K such that the fixed field H of the kernel is a CM field, and such that at least one prime above p in H^+ splits in H.*

Proof. Let F be the fixed field of ϵ. Let $\widetilde{F}$ be the Galois closure of F. The group $\mathrm{Gal}(\widetilde{F}/K)$ is an elementary abelian 2-group. The fixed fields of the conjugate representations $\epsilon_{K,g}$ are the conjugate fields F^g.

Let V_λ be an irreducible constituent of $\mathrm{Ind}_H^G \epsilon$ on which c acts by -1. As in the proof of Lemma 4.9, the representation $\mathrm{Res}_H V_\lambda$ contains every conjugate of ϵ. Thus $\mathrm{Gal}(\widetilde{F}/K)$ acts faithfully on V_λ. The action of G on V_λ factors through a quotient in which c is central and non-trivial. Thus the fixed field is a Galois CM field E with corresponding Galois totally real subfield E^+. The faithful action of $\mathrm{Gal}(\widetilde{F}/K)$ implies that $E.K = \widetilde{F}$. Let H^+ be $E^+ \cap K$. We break the remainder of the proof into two cases depending on whether $E \cap K = H^+$ or not.

1. Suppose that $K \cap E = H^+$. Then we have the following diagram of fields:

$$\begin{tikzcd}
K \arrow[r, dash] \arrow[d, dash] & K.E^+ \arrow[r, dash] \arrow[d, dash] & \widetilde{F} = K.E \arrow[d, dash] \\
H^+ = K \cap E \arrow[r, dash] & E^+ \arrow[r, dash] & E
\end{tikzcd}.$$

   It follows that there exists a canonical isomorphism $\mathrm{Gal}(\widetilde{F}/K) \simeq \mathrm{Gal}(E/H^+)$, and in particular the field F descends to a quadratic extension H/H^+. Every extension between E and H^+ is either totally complex or totally real, and is totally real if and only if it is contained in E^+. If H were contained in E^+, then F would be contained in $K.E^+$, which is Galois over $\mathbf{Q}$ since both K and E^+ are. Yet the Galois closure of F is $\widetilde{F}$, and thus H/H^+ is a CM extension. Suppose that all the primes above p in H^+ are inert in H. Then since p is completely split in K/H^+ it follows that all primes above p in K are inert in F. The same argument applies *mutatis mutandis* to all the conjugates of F. Yet this contradicts the fact that ϕ acts with odd order on W_g for some g.

2. Suppose that $E \cap K$ has degree two over $H^+ = E^+ \cap K$. Then necessarily it defines a CM extension. We have the following diagram of fields:

$$\begin{tikzcd}
K \arrow[r, dash] \arrow[d, dash] & \widetilde{F} = K.E \arrow[d, dash] \\
K \cap E \arrow[r, dash] \arrow[d, dash] & E \arrow[d, dash] \\
H^+ = K \cap E^+ \arrow[r, dash] & E^+
\end{tikzcd}.$$

   It follows that there exists a canonical isomorphism $\mathrm{Gal}(\widetilde{F}/K) \simeq \mathrm{Gal}(E^+/H^+)$, and thus the field F descends to a totally real quadratic extension H^{++} of H^+. Since H^{++}/H^+ is totally real and $K \cap E/H^+$ is CM, the compositum $(K \cap E).H^{++}$ contains a third quadratic extension H/H^+, which must be CM. Moreover by construction we see that $K.H = F$. Thus H descends F. Arguing as in case one we deduce that at least one prime above H^+ splits in H.

$\square$

6.3 Varying the involution

It remains to prove Lemma 6.7 which, for the convenience of the reader, we restate here.

Lemma 6.9. *Suppose that the action of ϕ on W_g is through an element of order two for all g, and that G admits at least one complex conjugation $c \notin N$. Then there exists a choice of involution ι such that the projection $\pi(\mathrm{Ind}_H^G \epsilon, v_\lambda)$ of v_λ to $\mathrm{Ind}_H^G \epsilon$ is not a -1 eigenvector for all conjugates c.*

Proof. Recall that the definition of ι involved for each $g \in G/N$ a choice of element $\psi \in \Gamma$ of order two distinct from (and therefore not commuting with) ϕ. Consider any such choice. There is an isomorphism of vector spaces

$$\Pi(\iota) : \bigoplus_{G/H} E \to \mathrm{Ind}_H^G \epsilon$$

defined as follows. The space Y^0 projects onto both spaces via the maps:

$$Y^0 \to Y^0/Y^{00} = \bigoplus_{G/H} E, \qquad Y^0 \subseteq Y \to \mathrm{Ind}_H^G \epsilon$$

respectively (the surjection of the second map follows from Lemma 5.14 part two). These maps identify Y^0 with the direct product of these spaces (exactly as in Sublemma 3 of Lemma 5.14). Now we define $\Pi(\iota)$ by taking $Y' \cap Y^0 = Y^{\iota=-1} \cap Y^0 \subseteq Y^0$ to be the graph of $\Pi(\iota)$. Explicitly the map $\Pi(\iota)$ decomposes into a direct product of the maps η_ψ of Lemma 5.14 for each $g \in G/H$. On the other hand, by construction, to compute $\Pi(\iota)(\lambda)$ one lifts it to Y^0, intersects with Y' and then projects down to Ind_H^G, and thus the image of λ is exactly $w_\lambda := \pi(\mathrm{Ind}_H^G \epsilon, v_\lambda)$.

Let $\gamma \in G/H$ be an element such that the γ-component of $w_\lambda \neq 0$. Suppose we vary the choice of element ξ of order two in the construction of ι for the $g \in G/N$ corresponding to γ. Then we obtain a new vector v'_λ and a new projection

$$w'_\lambda := \pi(\mathrm{Ind}_H^G \epsilon, v'_\lambda) = \Pi(\iota')(\lambda).$$

By Lemma 5.14, $\Pi(\iota)$ differs from $\Pi(\iota')$ only for those g of the form γn for $n \in N$, and *does* differ for those g (since $\eta_\psi \neq \eta_\xi$). Thus the only non-zero components of $(w_\lambda - w'_\lambda) \in \mathrm{Ind}_H^G \epsilon$ are contained in $g \in G/H$ of the form $g = \gamma n$ with $n \in N$, and $(w_\lambda - w'_\lambda)$ *does* have a non-zero component at g. If w_λ and w'_λ both generate representations on which c acts by -1, then they are both -1 eigenvectors for gcg^{-1} for any g, and in particular for $\gamma c \gamma^{-1}$. Yet then $\gamma c \gamma^{-1}(w_\lambda - w'_\lambda) = (w'_\lambda - w_\lambda)$ has a non-zero component at $(\gamma c \gamma^{-1})\gamma = \gamma c$, and hence $c \in N$. This is a contradiction, and thus at least one of w_λ or w'_λ generates a representation V_λ on which c does not act centrally by -1. $\square$

6.4 Proof of Proposition 6.5

To summarize, in Definition 6.3, we have constructed (under our hypotheses) a subspace $Y'(\lambda) \subset \Lambda$ with the following properties:

1. $Y'(\lambda)$ admits a decomposition $Y'(\lambda) = \bigoplus S_i(\lambda) \otimes T_i^* \subseteq \; Y' = \bigoplus S_i \otimes T_i^* \subseteq \Lambda$,
2. $\dim(S_i(\lambda)) \geq \dim(V_i | c = -1)$.

We prove that this contradicts the Strong Leopoldt Conjecture. After shrinking the spaces $S_i(\lambda)$, if necessary, we may assume that $\dim(S_i(\lambda)) = \dim(V_i|c = -1)$. Because $Y'(\lambda)$ admits a direct sum decomposition as above, from the discussion preceding Lemma A.15, we infer the existence of a G-submodule $N \subset \Lambda$ such that

$$\mathrm{Hom}_G(\Lambda/N, Y) = Y'(\lambda).$$

Furthermore, from the final claim of Lemma A.15, we may deduce the possible structures of N as an abstract G-module from the dimension formula $\dim(\mathrm{Hom}_G(N, V_i)) = \dim(V_i) - \dim(S_i(\lambda)) =$

$\dim(V_i|c=1)$. Assuming the classical conjecture of Leopoldt, one has a decomposition

$$I = I_{\mathrm{Global}} \simeq \bigoplus_{\mathrm{Irr}(G) \neq E} V_i^{\dim(V_i|c=1)}$$

of G-modules, where the sum runs over all *non-trivial* irreducible representations of G (see, for example, [22, Ch IX, §4]). It follows that we may assume that $I \simeq N$ as G-modules. Hence, by Lemma A.14,

$$\dim(\mathrm{Hom}_G(\Lambda/I, Y; Z)) \leq \dim(\mathrm{Hom}_G(\Lambda/N, Y; Z)) := Y'(\lambda) \cap Z$$

for any D-rational subspace Z of Y. Applying this lemma to $Z = Y^{00}(\lambda)$, we deduce, using Lemma 6.1, that

$$1 \leq \dim(\mathrm{Hom}_G(\Lambda/I, Y; Y^{00}(\lambda))) \leq Y'(\lambda) \cap Y^{00}(\lambda) = 0,$$

which is a contradiction.

7 Ordinary families with non-parallel weights

7.1 A result in the spirit of Ramakrishna

Let p be prime, and let $K/\mathbf{Q}$ be an imaginary quadratic field in which p splits. Let $\mathbf{F}$ be a finite field of characteristic p, and let χ denote the cyclotomic character. Let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbf{F})$ denote an absolutely irreducible continuous Galois representation, and suppose, moreover, that $\overline{\rho}$ is ordinary at the primes dividing p. The global Euler characteristic formula implies that the universal nearly ordinary deformation ring of $\overline{\rho}$ with fixed determinant has dimension ≥ 1. By allowing extra primes of ramification, one may construct infinitely many one parameter nearly ordinary deformations of $\overline{\rho}$. Our conjecture (1.3) predicts that these families contain infinitely many automorphic Galois representations if and only if they are CM or arise from base change. Since many families are not of this form (for example, if $\overline{\rho}$ is neither Galois invariant nor induced from a quadratic character of a CM field, then no such family is of that form), one expects to find many such one-parameter families with few automorphic points. In this section we prove that for many $\overline{\rho}$, there *exist* nearly ordinary families deforming $\overline{\rho}$ with only finitely many specializations ρ_x with equal Hodge-Tate weights. Since all cuspidal automorphic forms for $\mathrm{GL}(2)_{/K}$ of cohomological type have parallel weight, only finitely many specializations of ρ can therefore possibly be the (conjectural) Galois representation attached to such an automorphic form (for a more explicated version of what we are "conjecturing" about these forms, see section 8.1).

Let χ denote the cyclotomic character. If Σ denotes a (possibly infinite) set of places of K, let G_Σ denote the Galois group of the maximal extension of K unramified outside Σ.

Theorem 7.1. *Let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbf{F})$ be a continuous (absolutely irreducible) Galois representation, and assume the following:*

1. *The image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbf{F})$, and $p \geq 5$.*
2. *If $v|p$, then $\overline{\rho}$ is nearly ordinary at v, and takes the shape: $\rho|_{I_v} = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}$, where $\psi \neq 1, \chi^{-1}$, and $*$ is très ramifiée (see [37]) if $\psi = \chi$.*

3. *If $v \nmid p$ and $\overline{\rho}$ is ramified at v, then $H^2(G_v, \mathrm{ad}^0(\overline{\rho})) = 0$.*

Then there exists a Galois representation: $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(W(\mathbf{F})[[T]])$ lifting $\overline{\rho}$ such that:

1. *The image of ρ contains $\mathrm{SL}_2(W(\mathbf{F})[[T]])$.*
2. *ρ is unramified outside some finite set of primes Σ. If $v \in \Sigma$ and $v \nmid p$, then $\rho|D_v$ is potentially semistable; if $v|p$ then $\rho|D_v$ is nearly ordinary.*
3. *Only finitely many specializations of ρ have parallel weight.*

Proof. The proof of this theorem relies on techniques developed by Ramakrishna [32, 33]. Hypotheses 3 is almost certainly superfluous. For reasons of space, and to avoid repetition, we make no attempt in this section to be independent of [32]. In particular, we feel free to refer to [32] for key arguments, and try to keep our notation as consistent with [32] as possible.

If M is a finite Galois module over K, recall that M^* denotes the Cartier dual of M. Let M be of order a power of p, let V be a finite set of primes of K including all primes dividing p and ∞ as well as all primes with respect to which M is ramified. Let S denote the minimal such V, i.e., the set of primes dividing p, ∞ and ramified primes for M. For $i = 1, 2$ let

$$Ш^i_V(M) := \ker \left\{ H^i(G_V, M) \to \prod_{v \notin V} H^i(G_v, M) \right\}.$$

The group $Ш^1_V(M)$ is dual to $Ш^2_V(M^*)$.

Recall the global Euler characteristic formula:

$$\frac{\#H^0(G_V, M)\#H^2(G_V, M)}{\#H^1(G_V, M)} = \frac{\prod_{v|\infty} \#H^0(G_v, M)}{\#M^{[K:\mathbf{Q}]}},$$

where the product runs over all infinite places of K.

Let S be as above, and put $r = \dim\ Ш^1_S(\mathrm{ad}^0(\overline{\rho})^*)$.

Lemma 7.2. $\dim H^1(G_S, \mathrm{ad}^0(\overline{\rho})) = r + 3$.

Proof. Our assumptions on $\overline{\rho}|I_v$ together with assumption 3 imply that $H^2(G_v, \mathrm{ad}^0(\overline{\rho})) = 0$ for all $v \in S$. Thus $H^2(G_S, \mathrm{ad}^0(\overline{\rho})) \simeq Ш^2_S(\mathrm{ad}^0(\overline{\rho}))$, and hence

$$\dim H^2(G_S, \mathrm{ad}^0(\overline{\rho})) = \dim\ Ш^1_S(\mathrm{ad}^0(\overline{\rho})^*) = r.$$

Since K is an imaginary quadratic field, there is exactly one place at infinity. Since (by the isomorphism above) $\dim H^2(G_S, \mathrm{ad}^0(\overline{\rho})) = r$, the Global Euler characteristic Formula for $M := \mathrm{ad}^0(\overline{\rho})$ gives that

$$\dim H^1(G_S, \mathrm{ad}^0(\overline{\rho})) = r + \dim H^0(G_S, \mathrm{ad}^0(\overline{\rho})) + 2 \cdot \dim(\mathrm{ad}^0(\overline{\rho})) - \dim(H^0(G_\infty, \mathrm{ad}^0(\overline{\rho}))) = r + 3,$$

where we use the absolute irreducibility of $\overline{\rho}$ and the fact that G_∞ is trivial. □

If $v|p$, then $H^2(G_v, \mathrm{ad}^0(\overline{\rho})) = 0$, by assumption 3. The local Euler characteristic formula implies that

$$\#H^1(G_v, \mathrm{ad}^0(\overline{\rho})) = \#H^2(G_v, \mathrm{ad}^0(\overline{\rho})) \cdot \#H^0(G_v, \mathrm{ad}^0(\overline{\rho})) \cdot \#\mathrm{ad}^0(\overline{\rho}),$$

and thus $\dim H^1(G_v, \mathrm{ad}^0(\overline{\rho})) = 3$ (by assumption 2). Let $\mathcal{N}_v \subset H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$ for $v|p$ denote the 2-dimensional subspace corresponding to nearly ordinary deformations of $\overline{\rho}$. Note that our $\mathcal{N}_v$ is different than Ramakrishna's, where it is defined as the 1-dimensional subspace of $H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$ corresponding to *ordinary* deformations. Over $\mathbf{Q}$, the difference between ordinary and nearly ordinary deformations is slight, and amounts to fixing the determinant and replacing representations over $W(\mathbf{F})[[T]]$ by representations over $W(\mathbf{F})$. However, over K, this flexibility will be crucial. If we insist on an unramified quotient locally (at each $v|p$) and also fix the determinant, the relevant ordinary deformation ring will have expected relative dimension -1 over $W(\mathbf{F})$, and so we would not expect to find characteristic zero lifts in general. If $v \nmid p$ and $v \in S$, we take $\mathcal{N}_v = H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$, as in [32] p.144 (Recall that $H^2(G_v, \mathrm{ad}^0(\overline{\rho})) = 0$, by assumption).

We shall consider primes $\mathfrak{q} \notin S$ such that up to twist $\overline{\rho}|_{G_\mathfrak{q}} = \begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix}$, and $N(\mathfrak{q}) \not\equiv \pm 1 \mod p$. For these primes $\dim H^2(G_v, \mathrm{ad}^0(\overline{\rho})) = 1$, and $\mathcal{N}_v \subset H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$ is the codimension one subspace corresponding to deformations of the form:

$$\rho|_{G_\mathfrak{q}} = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}.$$

We call such primes auxiliary primes for $\overline{\rho}$.

Let $\{f_1, \ldots, f_{r+3}\}$ denote a basis for $H^1(G_S, \mathrm{ad}^0(\overline{\rho}))$. We recall from [32] p.138 the definition of some extensions of K. Let $\mathbf{K}$ denote the compositum $K(\mathrm{ad}^0(\overline{\rho})) \cdot K(\mu_p)$. Then $K \subset \mathbf{K} \subset K(\bar{\rho})$. If $f_i \in H^1(G_S, \mathrm{ad}^0(\overline{\rho}))$, let $\mathbf{L}_i/\mathbf{K}$ denote the extension with $\mathbf{L}_i$ the fixed field of the kernel of the morphism $\phi(f_i) : \mathrm{Gal}(\overline{K}/\mathbf{K}) \to \mathrm{ad}^0(\overline{\rho})$, where ϕ is the composition:

$$\phi : H^1(G_S, \mathrm{ad}^0(\overline{\rho})) \to H^1(\mathrm{Gal}(\overline{K}/\mathbf{K}), \mathrm{ad}^0(\overline{\rho})) = \mathrm{Hom}(\mathrm{Gal}(\overline{K}/\mathbf{K}), \mathrm{ad}^0(\overline{\rho})).$$

We have avoided issues arising from fields of definition (of concern to Ramakrishna in [32], p. 140) since, by assumption one, the image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbf{F})$. Let $\mathbf{L}$ denote the compositum of all the $\mathbf{L}_i$, and we have a natural isomorphism

$$\mathrm{Gal}(\mathbf{L}/\mathbf{K}) \simeq \prod_{i=1}^{r+3} \mathrm{Gal}(\mathbf{L}_i/\mathbf{K})$$

(cf. [32], p.141).

We recall the construction of some elements in $\mathrm{Gal}(\mathbf{L}/K)$ from [32] p.141–142. Let $c \in \mathrm{Gal}(\mathbf{K}/K)$ denote an element possessing a lift $\tilde{c} \in \mathrm{Gal}(K(\overline{\rho})/K)$ such that $\overline{\rho}(\tilde{c})$ has distinct eigenvalues with ratio $t \not\equiv \pm 1 \mod p$ and whose projection to $\mathrm{Gal}(K(\mu_p)/K)$ is t. Since the image $\overline{\rho}(\tilde{c})$ is defined up to twist, the ratio of its eigenvalues is independent of the lift $\tilde{c}$; i.e., is dependent only on c. Let

$$\alpha_i \in \mathrm{Gal}(\mathbf{L}/\mathbf{K}) \simeq \prod_{j=1}^{r+3} \mathrm{Gal}(\mathbf{L}_j/\mathbf{K})$$

denote the (unique up to scalar) element which, when expressed as an $r+3$-tuple in $\prod_{j=1}^{r+3} \mathrm{Gal}(\mathbf{L}_j/\mathbf{K})$ has all entries trivial except for the i-th at which its entry is given by a nonzero element $\alpha_i \in \mathrm{Gal}(\mathbf{L}_i/\mathbf{K})$ on which a lifting of c to $\mathrm{Gal}(\mathbf{L}_i/K)$ acts trivially by conjugation. (cf. [32] Fact 12).

Let T_i denote the Cebotarev class of primes in K corresponding to the element $\alpha_i \rtimes c \in \mathrm{Gal}(\mathbf{L}/K)$. We have implicitly used the assumption that $p \geq 5$ to deduce that $\mathrm{Gal}(\mathbf{L}/K)$ is a semidirect product of $\mathrm{Gal}(\mathbf{K}/K)$ by $\mathrm{Gal}(\mathbf{L}/\mathbf{K})$. Using the element c constructed above, the fact that $p \geq 5$, and that (when $p = 5$) the image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbf{F})$, one may establish the following (cf. [32], Fact 16, p.143):

Lemma 7.3. *There is a set $Q = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$ of auxiliary primes $\mathfrak{q}_i \notin S$ such that:*

1. *$\mathfrak{q}_i$ is unramified in $K(\overline{\rho})$, and is an auxiliary prime for $\overline{\rho}$.*
2. *$Ш^1_{S\cup Q}(\mathrm{ad}^0(\overline{\rho})^*) = 0$ and $Ш^2_{S\cup Q}(\mathrm{ad}^0(\overline{\rho})) = 0$.*
3. *$f_i|G_{\mathfrak{q}_j} = 0$ for $i \neq j$, $1 \leq i \leq r+3$ and $1 \leq j \leq r$.*
4. *$f_j|G_{\mathfrak{q}_j} \notin \mathcal{N}_{\mathfrak{q}_j}$ for $1 \leq j \leq r$.*
5. *The inflation map $H^1(G_S, \mathrm{ad}^0(\overline{\rho})) \to H^1(G_{S\cup Q}, \mathrm{ad}^0(\overline{\rho}))$ is an isomorphism.*

Proof. See [32], p.143, and [33], p.556. Note that (in our context) the parameter s is assumed to be zero. Of use in verifying that the same argument works in our context (in particular [32], p.143, the last paragraph of the proof) is the following identity:

$$\dim H^0(G_\infty, \mathrm{ad}^0(\overline{\rho})) = 3 = \dim H^1(G_S, \mathrm{ad}^0(\overline{\rho})) - r.$$

□

Let us consider the map:

$$\Phi : H^1(G_S, \mathrm{ad}^0(\overline{\rho})) \to \prod_{v\in S} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v = \prod_{v|p} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v.$$

Let d denote the dimension of $\ker(\Phi)$. Since the source has dimension $r+3$ and the target has dimension 2, it follows that $d \geq r+1$. Let $\{f_1, \ldots, f_d\}$ denote a basis of this kernel, and augment this to a basis $\{f_1, \ldots, f_{r+3}\}$ of $H^1(G_S, \mathrm{ad}^0(\overline{\rho}))$. As in [32], p.145, applying Lemma 7.3 to this ordering of f_i, we may assume that for $1 \leq i \leq r$ the element f_i restricts to zero in $H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$ for all $v \in S \cup Q$ except $\mathfrak{q}_i$, for which $f_i|G_{\mathfrak{q}_i} \notin \mathcal{N}_{\mathfrak{q}_i}$. We now consider the elements $\{f_{r+1}, \ldots, f_d\}$. Here we note one important difference from Ramakrishna [32]; since $d \geq r+1$ this set is necessarily non-empty.

Lemma 7.4. *For $i = r+1, \ldots, d$ the restriction map:*

$$\theta : H^1(G_{S\cup Q\cup T_i}, \mathrm{ad}^0(\overline{\rho})) \to \prod_{v\in S} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$$

is surjective. For $i = r+1, \ldots d-1$ there exist primes $\wp_i \in T_i$ such that the map:

$$H^1(G_{S\cup Q\cup\{\wp_{r+1},\ldots,\wp_{d-1}\}}, \mathrm{ad}^0(\overline{\rho})) \to \prod_{v\in S} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v$$

is surjective.

Proof. The first part of this Lemma is [32], Proposition 10, p.146, and the same proof applies. The second part follows exactly as in [32], Lemma 14, p.147, except that only $d-r-1$ primes are required. Explicitly, the target has dimension two, and the image of $H^1(G_S, \mathrm{ad}^0(\overline{\rho}))$ has dimension $r+3-d$, and $2-(r+3-d) = d-r-1$. $\square$

Remark: The reason why the numerology differs from that in Ramakrishna is that we are not fixing the determinant. Note, however, that imposing a determinant condition would impose *two* separate conditions at each $v|p$.

Let T denote the set of primes $\{\wp_{r+1}, \ldots, \wp_{d-1}\}$. That is, T is a finite set consisting of a single prime in T_i for $i = r+1, \ldots, d-1$. Combining the above results and constructions we arrive at the following:

Lemma 7.5. *The map*

$$H^1(G_{S\cup Q\cup T}, \mathrm{ad}^0(\overline{\rho})) \to \bigoplus_{v\in S\cup Q\cup T} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v$$

is surjective, and the kernel is one dimensional.

Proof. The source has dimension $r+3+(d-r-1) = d+2$, and the target has dimension $2+(d-1) = d+1$. The argument proceeds as in [32], Lemma 16, p.149, except now one has no control over the image of the cocycle f_d, and one may only conclude that the kernel is at most one dimensional. By a dimension count this implies the lemma. $\square$

Thus we arrive at the following situation. Let $\Sigma = S \cup Q \cup T$. Then $Ш^2_{\Sigma}(\mathrm{ad}^0(\overline{\rho})) = 0$ by Tate-Poitou duality (cf. [32], p.151) and the fact (Lemma 7.3) that $Ш^1_{S\cup Q}(\mathrm{ad}^0(\overline{\rho})^*) = 0$. Thus the map:

$$H^2(G_\Sigma, \mathrm{ad}^0(\overline{\rho})) \to \prod_{v\in\Sigma} H^2(G_v, \mathrm{ad}^0(\overline{\rho}))$$

is an isomorphism. In other words, all obstructions to deformation problems are local. On the other hand, we have a surjection

$$H^1(G_\Sigma, \mathrm{ad}^0(\overline{\rho})) \to \prod_{v\in\Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v$$

with one dimensional kernel (the source has dimension $d+2$ and the target has dimension $d+1$). Ramakrishna's construction identifies the local deformation rings (with respect to the subspaces $\mathcal{N}_v$) as power series rings, and thus smooth quotients of the unrestricted local deformation rings. Thus by the remark above there are no obstructions to lifting representations which are locally in $\mathcal{N}_v$, and thus by the dimension calculation above one has the following:

Lemma 7.6. *Let R_Σ denote the universal deformation ring of $\overline{\rho}$ subject to the following constraints:*

1. *The determinant of ρ is some given lift of* $\det(\overline{\rho})$ *to* $W(\mathbf{F})$.
2. *ρ is nearly ordinary at $v|p$.*
3. *ρ is minimally ramified at $v|S$ for $v \nmid p$.*
4. *$\rho|G_{\mathfrak{q}}$ takes the shape* $\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$ *for auxiliary primes* $\mathfrak{q} \in \Sigma \setminus S$.

Then $R_\Sigma \simeq W(\mathbf{F})[[T]]$*, and the image of the corresponding universal deformation*

$$\rho_\Sigma^{univ} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(W(\mathbf{F})[[T]])$$

contains $\mathrm{SL}_2(W(\mathbf{F})[[T]])$.

This representation satisfies all the conditions required by Theorem 7.1 except (possibly) the claim about specializations with parallel weight. To resolve this, we shall construct one more auxiliary prime $\wp$. Let $f_d \in H^1(G_\Sigma, \mathrm{ad}^0(\overline{\rho}))$ denote a nontrivial element in the kernel of the reduction map

$$H^1(G_\Sigma, \mathrm{ad}^0(\overline{\rho})) \to \prod_{v \in \Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v.$$

Then the restriction of f_d lands in

$$\prod_{v|\Sigma} \mathcal{N}_v.$$

We may detect the infinitesimal Hodge-Tate weights (or at least their reduction modulo p) by the analog of the characteristic zero construction in section 2. In particular, the infinitesimal Hodge-Tate weights are obtained by taking the image of f_d under the projection:

$$\omega : \prod_{v|\Sigma} \mathcal{N}_v \to \prod_{v|p} \mathcal{N}_v \to \prod_{v|p} \mathbf{F}.$$

A priori, three possible cases can occur:

1. The image of f_d in $\mathbf{F} \oplus \mathbf{F}$ is zero[2].
2. The image of f_d in $\mathbf{F} \oplus \mathbf{F}$ is non-zero but *diagonal*, i.e., a multiple of $(1,1)$.
3. The image of f_d in $\mathbf{F} \oplus \mathbf{F}$ is non-zero but not diagonal, i.e., not a multiple of $(1,1)$.

If the family ρ_Σ^{univ} has parallel weight, then the infinitesimal Hodge-Tate weights at any specialization will be (a multiple of) $(1,1)$, and thus we will be in case 2. If the family ρ_Σ^{univ} has constant weight, then we shall be in case 1, and if the family has non-constant but non-parallel weight, case 3 occurs. Thus we may assume we are in the first two cases. To reduce to the third case, we introduce a final auxiliary prime $\wp_d$. By construction, $\{f_1, \ldots, f_{d-1}, f_{d+1}, f_{d+2}\}$ maps isomorphically onto $\bigoplus_{v|\Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v$. We choose a splitting

$$\bigoplus_{v|\Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho})) = \left(\bigoplus_{v|\Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v\right) \oplus \left(\bigoplus_{v|\Sigma} \mathcal{N}_v\right)$$

such that the first factor is generated by $f_1, \ldots, f_{d-1}, f_{d+1}, f_{d+2}$. Recall that T_i denotes the primes not in $S \cup Q$ whose Frobenius element is in the conjugacy class $\alpha_i \rtimes c \in \mathrm{Gal}(\mathbf{L}/K)$.

[2] This case presumably never occurs, since it implies that the existence of a positive dimensional family of Global representations with constant Hodge-Tate weights.

Lemma 7.7. *The restriction map*

$$H^1(G_{\Sigma\cup T_d}, \mathrm{ad}^0(\overline{\rho})) \to \left(\bigoplus_{v\in\Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))\right)$$

is surjective.

Proof. The proof is very similar to Ramakrishna's proof of Proposition 10, p.146. This proof is, in effect, a calculation using Poitou-Tate duality. Exactly as in [32], p.146, we consider the restriction maps $\tilde{\theta}$ and $\tilde{\vartheta}$ (this notation defined exactly as in *loc. cit.*). It then suffices to prove that the annihilator of $\left(P^1_{T_d}(\mathrm{ad}^0(\overline{\rho})) + \mathrm{Image}(\tilde{\theta})\right)$ is trivial. For this, it suffices to show that the set $\{\tilde{\vartheta}(x)\}$ for $x \in H^1(G_{\Sigma\cup T_d}, \mathrm{ad}^0(\overline{\rho})^*)$ such that $x|G_v = 0$ for all $v \in T_d$ is trivial. Ramakrishna's argument assumes that $x|G_v = 0$ for all $v \in Q \cup T_d$, although he never uses that $x|G_v = 0$ for $v \in Q$. Thus θ surjects onto

$$\frac{P^1_{\Sigma\cup T_d}(\mathrm{ad}^0(\overline{\rho}))}{P^1_{T_d}(\mathrm{ad}^0(\overline{\rho}))} = \bigoplus_{v|\Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho})).$$

We remark that the same argument proves surjectivity of the reduction map with Σ replaced by $\Sigma \cup \widetilde{T}$, for any finite subset $\widetilde{T} \subset T_d$, although we do not use this fact. $\square$

Returning to our argument, we conclude from Lemma 7.7 that the map

$$H^1(G_{\Sigma\cup T_d}, \mathrm{ad}^0(\overline{\rho})) \to \bigoplus_{v|\Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho})) \to \bigoplus_{v|\Sigma} \mathcal{N}_v \xrightarrow{\omega} \bigoplus_{v|p} \mathbf{F}$$

is surjective, where this projection map is obtained from the splitting above (constructed from the cocycles $\{f_1, \ldots, f_{d-1}, f_{d+1}, f_{d+2}\}$). Thus there exists a auxiliary prime $\wp_d$ such that the image of the new cocycle f_{d+3} in $H^1(G_{\Sigma\cup\{\wp_d\}}, \mathrm{ad}^0(\overline{\rho}))$ projects to a non-diagonal subspace. Let $\Sigma' = \Sigma\cup\{\wp_d\}$. Exactly as previously, the map

$$H^1(G_{\Sigma'}, \mathrm{ad}^0(\overline{\rho})) \to \bigoplus_{v\in\Sigma'} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v$$

is surjective, with a one dimensional kernel. The kernel of the projection to the smaller space $\bigoplus_{v\in\Sigma} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v$ is two dimensional, and contains f_d. If f_{d+3} denotes the new cocycle in this kernel, then by construction the projection of f_{d+3} to $\mathbf{F} \oplus \mathbf{F}$ is not diagonal, since adjusting f_{d+3} by elements of $\{f_1, \ldots, f_{d-1}, f_{d+1}, f_{d+2}\}$ does not change this projection. Since $f_d|G_{\wp_d} \notin \mathcal{N}_{\wp_d}$, it follows that the kernel of the projection to $\bigoplus_{v\in\Sigma'} H^1(G_v, \mathrm{ad}^0(\overline{\rho}))/\mathcal{N}_v$ is generated by $f_{d+3} - \alpha \cdot f_d$ for some α, and thus the projection of this element to $\mathbf{F} \oplus \mathbf{F}$ is also not diagonal. This completes the proof. $\square$

7.2 Examples

Corollary 7.8. *Suppose that $\overline{\rho}$ satisfies the conditions of Theorem 7.1. Then there exists a set of primes Σ such that the if R_Σ is the universal nearly ordinary deformation ring of $\overline{\rho}$ unramified outside Σ then $\mathrm{Spec}(R_\Sigma)$ contains a one dimensional component with only finitely many representations of parallel weight.*

For example, one source of such $\overline{\rho}$ is the mod p representations associated to $\Delta = \sum_{n=1}^{\infty} \tau(n)q^n$ whenever $p \nmid \tau(p)$ and $11 \leq p \neq 23, 691$.

8 Complements

8.1 Galois representations associated to Automorphic Forms

Let $K/\mathbf{Q}$ be an imaginary quadratic field, and let π denote an automorphic representation of cohomological type for $\mathrm{GL}_2(\mathbf{A}_K)$. It is expected, and known in many cases, that associated to π there exists a compatible family of λ-adic representations $\{\rho_\lambda\}$ such that for $v \nmid N(\lambda)$, the representation $\rho_\lambda|_{D_v}$ corresponds (via the local Langlands correspondence) to π_v. One may further conjecture a compatibility between $\rho_\lambda|_{D_v}$ and π_v for $v|N(\lambda)$, in the sense of Fontaine's theory. The weakest form of such a conjecture is to ask that the representation $\rho|_{D_v}$ be Hodge-Tate of the weights "predicted" (by the general Langlands conjectures) from the weight of π. If π is *cuspidal*, then a result of Harder ([12], §III, p.59–75) implies that π has parallel weight, and thus the expected Hodge-Tate weights of ρ_λ at $v|N(\lambda)$ are of the form $[0, k-1]$, for some $k \in \mathbf{Z}$ that does not depend on v. For the cases in which the existence of the Galois representation has already been established (by Taylor et.al. [13, 39]), one obtains this weaker compatibility automatically by construction, as we now explain. The Galois representations associated to π are constructed from limits of representations arising from Siegel modular forms. Since Sen-Hodge-Tate weights are analytic functions of deformation space [36], it suffices to determine the Hodge-Tate weights attached to Galois representations of Siegel modular forms in *classical* weight (a, b), and in particular to show that the Hodge-Tate weights of such forms are $[0, a-1, b-2, a+b-3]$. This can be proven by a comparison between étale cohomology and de Rham cohomology on Siegel threefolds (see, for example, [30]). When we talk in this paper of classical automorphic (Galois) representations associated to automorphic forms over K, we refer to the (possibly conjectural) Galois representations satisfying the usual local Langlands compatibilities at primes $v \nmid N(\lambda)$, and the compatibility at $v \mid N(\lambda)$ between the automorphic weight of π and the Hodge-Tate weights of $\rho_\lambda|_{D_v}$. Thus, by fiat, all such representations have Hodge-Tate weights $[0, k-1]$ for $v|N(\lambda)$ and for some $k \in \mathbf{Z}$ independent of v.

8.2 Even two-dimensional Galois representations over Q

Let V be a two-dimensional vector space over E and let $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}_E(V)$ be an even continuous (irreducible) nearly ordinary Artin representation. Let $K/\mathbf{Q}$ be an imaginary quadratic field in which p splits. Restrict ρ to G_K and give it a nearly ordinary structure *compatible with the action of* $\mathrm{Gal}(K/\mathbf{Q})$. Equivalently, if $v, \bar{v}$ are the places above p, with decomposition groups D_v and $D_{\bar{v}}$ chosen so that they are conjugate to one another in $G_{\mathbf{Q}}$ (i.e., $D_{\bar{v}} = \gamma D_v \gamma^{-1}$ where γ is some element in $G_{\mathbf{Q}}$), the chosen lines $L_v, L_{\bar{v}} \subset V$ are related to each other by the formula $L_{\bar{v}} = \gamma L_v$.

The representation $\rho|_K$ admits infinitesimally classical deformations, with infinitesimal Hodge-Tate weights lying in the anti-diagonal subspace of infinitesimal weight space (Definition 2.13)

$$\{(e, -e) \mid e \in E\} \subset E \oplus E \ = \ \{\bigoplus_{v|p} \mathbf{Q}\} \otimes_{\mathbf{Q}} \mathcal{E}.$$

To prove this, note that $H^1(K, \mathrm{ad}^0(\rho))$ decomposes as $H^1(\mathbf{Q}, \mathrm{ad}^0(\rho)) \oplus H^1(\mathbf{Q}, \mathrm{ad}^0(\rho) \otimes \chi)$, where χ is the quadratic character associated to K. The first cohomology group records nearly ordinary deformations of ρ *over* $\mathbf{Q}$, while the second measures nearly ordinary deformations *over* K in the "anticyclotomic" direction. The fixed field of the kernel of $\mathrm{ad}^0(\rho)$ is totally real, for the following reason. Either complex conjugation acts as $+1$ in the representation ρ (in which case the kernel of ρ itself is totally real) or else it acts as the scalar -1 in the representation ρ and therefore acts as the identity in $\mathrm{ad}^0(\rho)$. The Euler characteristic formula then implies that

$$\dim(H^1(\mathbf{Q}, \mathrm{ad}^0(\rho))) \geq 0, \qquad \dim(H^1(\mathbf{Q}, \mathrm{ad}^0(\rho) \otimes \chi)) \geq 1,$$

with equality in either case following from the Leopoldt conjecture.

If $\rho|_K$ lies in a one-parameter (or many-parameter, for that matter) family of G_K-representations in which the classical automorphic representations (i.e., those Galois representations corresponding conjecturally, as in section 8.1, to classical automorphic form for $\mathrm{GL}(2)_{/K}$) are Zariski-dense, then the infinitesimal Hodge-Tate weights of this representation would be diagonal, i.e., would lie in the diagonal subspace

$$\{(e, e) \mid e \in E\} \subset E \oplus E \ = \ \{\bigoplus_{v|p} \mathbf{Q}\} \otimes_{\mathbf{Q}} \mathcal{E}.$$

Thus if we assume the Leopoldt conjecture (specifically, for the compositum of K and the fixed field of ρ), we deduce the following:

Theorem 8.1. *Let $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(E)$ be an even continuous (irreducible) nearly ordinary Artin representation. Let $K/\mathbf{Q}$ be an imaginary quadratic field in which p splits. Assume the Leopoldt conjecture for the fixed field of $\rho|_K$. Then the representation $\rho|_K$ does not deform to a one-parameter family of nearly ordinary Galois representations arising from classical automorphic forms over K.*

Note that the proof [5] of the Artin conjecture for certain *odd* representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ requires the following idea: one places the representation in question into a one parameter family of Galois representations in which the representations associated to classical ordinary modular forms are Zariski dense. The above theorem shows that even the first step in such a program for even representations (extending the base field such that ρ admits deformations to other classical automorphic forms) does not naïvely work.

Suppose that H^+ is a Galois extension in which p completely splits, and H/H^+ is a CM extension in which at least one prime above p in H^+ splits in H. Then there exist algebraic Hecke characters η of $\mathbf{A}_H^\times$ such that the Galois representations associated to their automorphic inductions on $\mathrm{GL}_2(\mathbf{A}_{H^+})$ are nearly ordinary and interpolate to form families of possibly lower dimension than the dimension of the (ambient) universal deformation space of nearly ordinary representations. The Hodge-Tate weights of these families are constant at the places corresponding to primes $v \mid p$ of H^+ that are inert in the extension H/H^+ — this is a manifestation of Lemma 6.7.

An example where the CM locus is non-trivial but strictly smaller than the nearly ordinary locus can easily be constructed: take a non-Galois extension of a real quadratic field H^+ such that p splits in H^+ and exactly one prime above p splits in H.

In both the "CM" and "Base Change" cases there exist classical families giving rise to the infinitesimally classical deformations.

In the "even" case of Theorem 1.2, the field $K^+ \subset K$ to which $\rho \otimes \chi$ descends does not need to be totally real. Let $K^+/\mathbf{Q}$ be a degree three field with signature $(1,1)$ in which p is completely split, and let K be its Galois closure. Let L/K^+ be an A_4 extension that is even at the real place, and such that $\mathrm{Gal}(M/\mathbf{Q}) = (A_4)^3 \ltimes S_3$, where M is the Galois closure of L. Suppose that the decomposition group at some place $v|p$ is generated by the element (u,u,u), where $u \in A_4$ has order three. Let ρ be the representation $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(E)$ associated to L/K. Then ρ admits an infinitesimally classical deformation.

8.3 Artin representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ of dimension n

The methods of the previous section can also be applied to Artin representations:

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(E)$$

when $n > 2$. For such a representation V, let $\mathrm{ad}^0(\rho) : G_{\mathbf{Q}} \to \mathrm{GL}_{n^2-1}(E)$ denote the corresponding adjoint representation. Let $G := \mathrm{Im}(\mathrm{ad}^0(\rho)) \subset \mathrm{GL}_{n^2-1}(E)$ and $\widetilde{G} = \mathrm{Im}(\rho) \subset \mathrm{GL}_n(E)$; the group $\widetilde{G}$ is a central extension of G. We assume that ρ is unramified at p, so the image of the decomposition group D at p is cyclic; we also assume that D acts with distinct eigenvalues on V, and write (having chosen once and for all an ordering)

$$V = \bigoplus_{i=1}^{n} L_i,$$

where L_i are D-invariant lines of V. Consider deformations which are nearly ordinary at p. Let $W = \mathrm{Hom}'(V,V)$ denote the hyperplane in $\mathrm{Hom}(V,V)$ consisting of endomorphisms of V of trace zero. The analog of the vector space $W^0 \subset W$ is $\mathrm{Ker}(W \to \bigoplus_{i<j} \mathrm{Hom}(L_i, L_j))$, which, after a suitable choice of basis, may be identified with upper triangular matrices of trace zero. There is a canonical map $\epsilon : W^0 \to \bigoplus \mathrm{Hom}(L_i, L_i) = E^n$, whose image is the hyperplane $H : \sum x_i = 0$. Let $H_{\mathbf{Q}}$ denote a model of the hyperplane H over $\mathbf{Q}$. The map ϵ induces, via the local class field theory, a map on infinitesimal deformations:

$$\omega : H^1_{\Sigma}(\mathbf{Q}, \mathrm{ad}^0(\rho)) \to H^1(D, H) = \mathrm{Hom}(D^{\mathrm{ab}}, H) \to \mathrm{Hom}(\mathbf{Q}_p^*, H) = H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{E},$$

where, as in section 2.3, $\mathcal{E} = \mathrm{Hom}(\mathbf{Z}_p^*, E)$. The definition of infinitesimal Hodge-Tate weights proceeds as in section 2.3.

Lemma 8.2. *If $W = \mathrm{ad}^0(V)$ is irreducible, then assuming the strong Leopoldt conjecture, ρ admits no infinitesimally classical deformations.*

Sketch. It suffices to construct a suitable space Y' of W which is "maximally skew" to the ordinary subspace, but still decomposes well with respect to the decomposition of W as a $G = \mathrm{Gal}(K/\mathbf{Q})$-representation, where K is the fixed field of $\mathrm{ad}^0(\rho)$. Yet, by assumption, W is irreducible, and so this is no condition at all. For example, let ι denote the involution of W such that $\iota\mathrm{Hom}(L_i, L_j) = \mathrm{Hom}(L_j, L_i)$. Then one may take $Y' = W^{\iota=-1}$. $\square$

Note that if ρ (equivalently, V) is self-dual and $n \geq 3$, then $W = \mathrm{ad}^0(V)$ is *never* irreducible, since $\mathrm{ad}(V) = V \otimes V^* \simeq V \otimes V$ decomposes into $\wedge^2 V$ and $\mathrm{Sym}^2(V)$, both of which have dimension > 1 if $n \geq 3$. The following is well known [9].

Lemma 8.3. *Let $\rho : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_3(E)$ be an absolutely irreducible Artin representation. Then the projective image of ρ is one of the following groups.*

1. *The simple alternating groups A_5 or A_6.*
2. *The simple linear group $\mathrm{PSL}_2(\mathbf{F}_7) = \mathrm{PSL}_3(\mathbf{F}_2)$ of order 168.*
3. *The "Hessian group" H_{216} of order 216.*
4. *The normal subgroup H_{72} of H_{216} of order 72.*
5. *The normal subgroup H_{36} of H_{72} of order 36.*
6. *An imprimitive subgroup; i.e., the semidirect product of $(\mathbf{Z}/3\mathbf{Z})$ and S_3 or A_3, or a subgroup of this group.*

Lemma 8.4. *If G lies in the list above, and ρ denotes a three dimensional representation of a central extension $\widetilde{G}$ of G with underlying vector space V, then $\mathrm{ad}^0(V)$ is irreducible except for the following cases:*

1. *ρ is the symmetric square up to twist of a two dimensional representation.*
2. *G is imprimitive.*
3. *$G = H_{36}$, in which case $\mathrm{ad}^0(\overline{\rho}) = M \oplus M^*$ where $\dim(M) = 4$.*

From this we may deduce the following:

Theorem 8.5. *Assume the strong Leopoldt conjecture. Suppose that $\rho : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_3(E)$ is continuous, nearly ordinary, and has absolutely irreducible finite image. Suppose moreover that ρ is unramified at p, that ρ is p-distinguished, and that the projective image of ρ is a primitive subgroup of $\mathrm{PGL}_3(E)$. Then if ρ admits infinitesimally classical deformations, then ρ is, up to twist, the symmetric square of a two dimensional representation.*

Proof. This follows from Lemma 8.2, and Lemmas 8.3 and 8.4, unless $G = H_{36}$. If $G = H_{36}$, one may proceed on a case by case analysis — each case corresponding to conjugacy class of Frob_p in G — and explicitly construct a suitable space Y' as in Lemma 8.2 that respects the decomposition $W = M \oplus M^*$ of W into the two 4-dimensional representations of G. $\square$

In general, as n becomes large, there are many groups which admit irreducible n-dimensional representations which are not self dual up to twist, and yet their corresponding adjoint representations (minus the identity) of dimension $n^2 - 1$ are not irreducible. Thus to prove a general result for n-dimensional representations would require an analog of our arguments for 2-dimensional representations over general fields[3].

[3]Of course, Lemma 8.2 may be applied to various sporadic examples when $n > 3$, for example, if V is either of the 11-dimensional representations of the Mathieu group M_{12} (which does occur as a Galois group over $\mathbf{Q}$ — see [26]), then $\mathrm{ad}^0(V)$ is the irreducible representation of M_{12} of dimension 120. Thus we may apply Lemma 8.2 for 11-dimensional representations ρ with fixed field $K/\mathbf{Q}$ of ρ satisfying the following: $\mathrm{Gal}(K/\mathbf{Q}) \simeq M_{12}$, $K/\mathbf{Q}$ unramified at p, $\mathrm{Gal}(K/\mathbf{Q}) = M_{12}$, and Frob_p has order 11.

8.4 Deformations of Characters: $\mathrm{GL}(1)_{/K}$

In this section we explain why the analog of our conjectures for characters (equivalently, automorphic forms for $\mathrm{GL}(1)_{/K}$) can be deduced from a simple application of class field theory. Namely, suppose that $\rho : \mathrm{Gal}(\overline{K}/K) \to E^*$ is the trivial representation, or indeed, any finite character. Suppose that K is Galois and that $G := \mathrm{Gal}(K/\mathbf{Q})$. Then the universal deformation ring R of ρ unramified away from p (and with no local condition at primes dividing p) may be described explicitly in terms of class field theory. In particular, the tangent space to R at ρ is equal to $\mathrm{Hom}_E(\Gamma_K, E)$, where Γ_K is the maximal $\mathbf{Z}_p$-extension of K unramified outside p. Identifying $\Gamma_K \otimes E$ as the quotient of $U_{\mathrm{Local}} \simeq \Lambda$ by (the image I of) U_{Global} as in section 5.4, we may write this as

$$\mathrm{Hom}_E(\Lambda/I, E) = \mathrm{Hom}_G(\Lambda/I, \mathrm{Ind}^G_{\langle 1 \rangle} E) = \mathrm{Hom}_G(\Lambda/I, \Lambda) = [I]\Lambda.$$

All one dimensional representations admit a trivial infinitesimal deformation arising from twisting via the cyclotomic character. The following result is the analog of Theorem 1.2

Lemma 8.6. *Assume that p splits completely in K, and suppose that K is Galois with Galois group G. If the trivial representation admits an infinitesimal deformation over $\mathbf{Q}$ which is not cyclotomic, then this deformation descends to a subfield H which is a CM field.*

Sketch. The rational infinitesimal weights correspond to the natural identification of Λ with $\Lambda_{\mathbf{Q}} \otimes E$. The existence of a rational infinitesimal deformation corresponds exactly to an inclusion $\eta \in [I]\Lambda$, where the vector space ηE is a rational subspace of Λ — equivalently, where $\eta \in \Lambda_{\mathbf{Q}} = \mathbf{Q}[G]$ (up to scalar). Without loss of generality we assume that $\eta \in \mathbf{Z}[G]$. If $\eta = \sum_{g \in G} g$ is the norm, then η *does* annihilate I (on the left and right) and so lies in the annihilator group $[I]\Lambda$ — such a choice corresponds to the cyclotomic deformation. Thus we assume that η is not a multiple of $N_{K/\mathbf{Q}}$, and in particular is not G-invariant. If we assume the strong Leopoldt conjecture, then as in section 6.4, there is an isomorphism of (right) G-modules:

$$[I]\Lambda \simeq E \oplus \bigoplus_{\mathrm{Irr}(G) \neq E} V_i^{\dim(V_i | c = -1)},$$

where the second product runs over all non-trivial irreducibles of G. If there was an equality $\dim(V_i | c = -1) = \dim(V_i)$ for some (any) representation V_i, the action of G on V_i would factor (faithfully) through $\mathrm{Gal}(H_i/\mathbf{Q})$ for some CM field H_i. Since, by assumption, $\eta \notin E$, the strong Leopoldt conjecture implies (as in the argument of section 6.4) that $\dim(V_i | c = -1) = \dim(V_i)$ for all representations V_i in which η projects non-trivially. It follows that η is divisible by $N_{K/H}$ (for some CM field H with is the compositum of the relevant CM fields H_i) and that the infinitesimal deformation descends to H. Thus the conclusion of the lemma follows from the strong Leopoldt conjecture. To prove the lemma without this assumption, we use the following result.

Sublemma 5. *If $\eta \in \mathbf{Z}[G]$ annihilates I, then it annihilates the global unit group U_{Global}.*

Proof. If the map $U_{\mathrm{Global}} \to I$ is injective (the Leopoldt conjecture) the result is obvious. Yet, in any circumstance, this map is G-equivariant, and thus the image of $(U_{\mathrm{Global}})\eta$ is $I\eta$. Since the map from the group of units $\otimes \mathbf{Q}$ (that is, the image of the map $U_{\mathrm{Global},\mathbf{Q}} \to I \subset U_{\mathrm{Local}}$) *is* injective, we deduce that $I\eta = 0$ if and only if $(U_{\mathrm{Global}})\eta = 0$. ∎

Since $U_{\mathrm{Global},\mathbf{Q}} \otimes \mathbf{C} \simeq \bigoplus_{\mathrm{Irr}(G) \neq E} V_i^{\dim(V_i|c=1)}$, we see that if an element $\eta \in \mathbf{Z}[G]$ other than the norm annihilates U_{Global}, then $\dim(V_i|c=1) = 0$ for some collection V_i. As above, we may deduce from this that the deformation descends to a CM field H. $\square$

This result could presumably be strengthened to include infinitesimal deformations over some number field F rather than $\mathbf{Q}$. For this we would have needed to show that U_{Global} was not annihilated by an element of $F[G]$, which would have required an appeal to Baker's results on linear forms in logarithms, as in Brumer's proof of Leopoldt's conjecture.

In the case of one dimensional representations there is no difficultly in establishing, by class field theory, an $R = \mathbf{T}$ theorem. The analog of Lemma 8.6 for automorphic representations (namely, that automorphic representations are not dense in $\mathrm{Spec}(\mathbf{T})$ for general K) follows from the fact that, up to finite characters, the only non-trivial algebraic Hecke characters arise from CM fields H. Indeed, an analysis of the infinity type shows that this question exactly reduces to determining the annihilator of $U_{\mathrm{Global},\mathbf{Q}}$ in $\mathbf{Z}[G]$.

8.5 Automorphic Forms for $\mathrm{GL}(2)_{/K}$

The goal of this subsection will be to sketch a proof of Theorem 1.1.

The group $\mathrm{GL}(2)_{/K}$ is associated to a PEL Shimura variety if and only if K is totally real. We assume that $K/\mathbf{Q}$ is an imaginary quadratic field in which p splits completely. Since we shall ultimately take $K = \mathbf{Q}(\sqrt{-2})$, we also assume that the class group of K is trivial. Let $\mathcal{O}$ be the localization of the ring of integers $\mathcal{O}_K$ of K at some prime above p. Let Γ_K be the torsion free part of $\mathcal{O}^\times$, and let $\Lambda = \mathcal{O}[[\Gamma_K]]$ be the Iwasawa algebra, which is abstractly isomorphic to $\mathcal{O}[[T_1, T_2]]$. Hida (page 29 of [15]) constructs a finitely generated Λ-module $\mathbf{H} := H^1_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})^*$. The right hand side of this equation is in the notation of *loc. cit.*; $Y(\Phi)$ is, in effect, the arithmetic quotient of the symmetric space for GL_2 over K of an appropriate level, and the standard Hecke operators act on this cohomology group. Following Hida, let $\mathbf{T}$ be the subring of endomorphisms of $\mathrm{End}(\mathbf{H})$ generated by Hecke operators. Then $\mathbf{T}$ is finitely generated over Λ. Hida shows in Theorem 5.2 of [15] that the support of $\mathbf{H}$ is some equidimensional space of codimension one in $\mathrm{Spec}(\Lambda)$.

Recall that by a "classical point" of $\mathrm{Spec}(\mathbf{T})$ we mean a $\mathbf{C}_p$-valued point corresponding to a homomorphism $\eta : \mathbf{T} \to \mathbf{C}_p$ for which there exists a classical automorphic eigenform (cf. subsection 8.1) whose Hecke eigenvalues are given by the $\mathbf{C}_p$-character η.

Let $K = \mathbf{Q}(\sqrt{-2})$, and let $\mathfrak{N} = 3 - 2\sqrt{-2}$. For the rest of this subsection we let $\mathbf{T}$ denote the nearly ordinary 3-adic Hida algebra of tame level $\mathfrak{N}$.

The affine scheme $\mathrm{Spec}(\mathbf{T} \otimes \mathbf{Q})$ is nonempty and contains at least one classical point, i.e., the point corresponding to the classical automorphic form f of weight $(2,2)$ of level $\Gamma_0(7 + \sqrt{-2})$ that is discussed in the following lemma, a proof of which will be given in the next subsection (Lemma 8.10).

Lemma 8.7. *Let $K = \mathbf{Q}(\sqrt{-2})$, let $\mathfrak{N} = (3 - 2\sqrt{-2})$, and let $\mathfrak{p} = (1 + \sqrt{-2})$. There exists a unique cuspidal eigenform f of weight $(2,2)$ and level $\Gamma_0(\mathfrak{N}\mathfrak{p}) = \Gamma_0(7 + \sqrt{-2})$ which is ordinary at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. There does not exist any cuspidal eigenform (ordinary or otherwise) of level $\Gamma_0(\mathfrak{N})$ and weight $(4,4)$.*

As mentioned above, Hida has shown that the nonempty affine scheme $\mathrm{Spec}(\mathbf{T})$ is of pure relative dimension one over $\mathrm{Spec}(\mathbf{Z}_p)$.

Theorem 8.8. *The scheme* $\mathrm{Spec}(\mathbf{T})$ *has only finitely many classical points.*

Remarks:

1. This answers in the negative a question raised in Richard Taylor's thesis ([38], Remark, p.124).
2. We expect, but have not proven, that the automorphic form f referred to in Lemma 8.7 corresponds to the elliptic curve
$$E : y^2 + \sqrt{-2}xy + y = x^3 + (\sqrt{-2} - 1)x^2 - \sqrt{-2}x$$
over $\mathbf{Q}(\sqrt{-2})$, which can be found in Cremona's tables [6, Table 3.3.3]. Presumably one could prove the association between f and E using the methods of [39].
3. We also expect that there is a two-dimensional Galois representation over $\mathbf{T}$ whose Frobenius-traces are equal to corresponding Hecke operators $\mathbf{T}$ (this correspondence being meant in the usual sense) but we have not constructed such a representation. If there were such a Galois representation, one effect of the above theorem is to show that *none* of the Galois representations over $\mathbf{C}_p$ obtained by specializing it to $\mathbf{C}_p$ via homomorphisms $\mathbf{T} \to \mathbf{C}_p$ is "limit-automorphic;" i.e., is the limit of a sequence of distinct classical modular Galois representations. This is in contrast to what is expected (and is often known) to be true over $\mathbf{Q}$, where *every* ordinary two-dimensional p-adic Galois representation of integral weight that is residually modular should be "limit-modular", in the sense that it should be obtainable as the limit of a sequence of modular ordinary representations whose weights tend to infinity.
4. In passing, we should note that it does not follow from the construction of Taylor et. al. [13, 39] that the Galois representation associated to an ordinary cuspidal modular form over K is ordinary in the Galois sense. However, in many cases this is known by a theorem of Urban [40].
5. A result analogous to Theorem 8.8 is established in [2] for ordinary families on $\mathrm{GL}(3)_{/\mathbf{Q}}$. In this setting, however, one does not have Hida's purity result on the dimension of $\mathrm{Spec}(\mathbf{T})$, and can only deduce the existence of a component of $\mathrm{Spec}(\mathbf{T})$ with finitely many classical points — *a priori* this component could have dimension zero. (See, however, work in progress of Ash and Stevens in which this result is established, essentially following Hida's methods.) One of the main motivations of this paper was to understand the examples of [2] from the perspective of Galois representations.

To prove Theorem 8.8, note that if $\mathrm{Spec}(\mathbf{T})$ contained infinitely many classical points, then since $\mathbf{T}$ is finitely generated over Λ it would follow that the support of $\mathbf{T}$ would include *every* classical parallel weight of $\mathrm{Spec}(\Lambda)$. In particular it would contain a classical point of weight $(4, 4)$. By Hida's control theorem [15, Thm 3.2], the specialization of $\mathbf{T}$ at a point of weight $(4, 4)$ would correspond to a classical automorphic form of level $\Gamma_0(3-2\sqrt{-2})$, which contradicts the statement of Lemma 8.7.

8.6 Modular Symbols

In order to compute explicit examples of Hida families (or the lack thereof) one needs a method of computing spaces of modular forms for imaginary quadratic fields. This is equivalent to computing group cohomology of congruence subgroups of $\mathrm{GL}_2(\mathcal{O}_K)$ with coefficients in some local system. In the case of weight $(2,2)$, the local system is trivial, and the problem reduces to a homology computation on the locally symmetric space $\mathcal{G}(K)\backslash\mathcal{G}(\mathbf{A}_K)/U = \mathcal{H}/\Gamma$, where $\mathcal{G} = \mathrm{Res}(\mathrm{GL}(2)_{/K})$, U is the appropriate compact open of $\mathrm{GL}_2(\mathbf{A}_K)$ containing the connected component of a maximal compact subgroup of $\mathrm{GL}_2(\mathbf{R})$, $\mathcal{H}$ is the hyperbolic upper half space, and Γ a finite index congruence subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$ (recall we are assuming that the class number of K is 1). One technique for doing this is to use modular symbols, and this was carried out by J. Cremona [6] for certain fields K of small discriminant and class number one. The only higher weight computations previously carried out are some direct computations of group cohomology in level one and weight $(2k,2)$ by C. Priplata [31]; however, as the only automorphic forms of this level (with $2k > 2$) are Eisenstein, the result was torsion. To compute in general weights (k,k) one must work with modular symbols of higher weight. The generalization of modular symbols to higher weight is completely standard and is analogous to the case of $\mathrm{GL}(2)_{/\mathbf{Q}}$. For our specific computations, we have decided to work with the field $K = \mathbf{Q}(\sqrt{-2})$. We choose this field because it is the simplest field in which 3 splits completely.

Let V be a two dimensional vector space over K. There is a natural action of $\mathrm{GL}_2(\mathcal{O})$ on V, and choosing a basis we may identify V with degree one homogeneous polynomials in $K[x,y]$. Let $S_k = \mathrm{Sym}^k(V)$ and let $\overline{S}_k$ denote S_k where the action of $\mathrm{GL}_2(\mathcal{O})$ is twisted by the automorphism of $\mathrm{Gal}(K/\mathbf{Q})$. Modular forms over K of weight (k,k) can be considered as sections of $H^1_{\mathrm{cusp}}(\mathcal{H}/\Gamma, \mathcal{S}_{k,k})$, where $\mathcal{S}_{k,k}$ is the local system associated to the representation $S_{k-2} \otimes \overline{S}_{k-2}$.

The following theorem is essentially due to Cremona[4] [6], although one should note that previous limited computations of these spaces were carried out in [11].

Theorem 8.9. *Let $K = \mathbf{Q}(\sqrt{-2})$, and let $\mathfrak{N} \subseteq \mathcal{O}_K$ be an ideal. Let M be the free K-module on the formal generators*

$$\left\{[c,P] \mid c \in \mathbf{P}^1(\mathcal{O}/\mathfrak{N}), P \in S_{k,k}\right\}.$$

There is an action of $\mathrm{GL}_2(\mathcal{O})$ on M given by $[c,P].h = [Ph, ch]$. Explicitly, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$P(x,y,x',y')\gamma = P(dx - cy, -bx + ay, \sigma(d)x' - \sigma(c)y', -\sigma(b)x' + \sigma(a)y'),$$

where $\mathrm{Gal}(K/\mathbf{Q}) = \langle\sigma\rangle$. The action of $\mathrm{GL}_2(\mathcal{O})$ on $\mathbf{P}^1(\mathcal{O}/\mathfrak{N})$ arises from the identification of this space with coset representatives of $\Gamma_0(\mathfrak{N})$ in $\mathrm{GL}_2(\mathcal{O})$. Let S, T, J and X denote the following elements of $\mathrm{GL}_2(\mathcal{O})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} \sqrt{-2} & 1 \\ 1 & 0 \end{pmatrix},$$

and let I be the identity. Let M_{rel} be the subspace of M generated by the subspaces:

$$(1+S)M, \quad (1 + (ST) + (ST)^2)M, \quad (I - J)M, \quad (1 + X + X^2 + X^3)M$$

for $m \in M$. Then $M/M_{rel} \simeq H^1(\mathcal{H}/\Gamma_0(\mathfrak{N}), \mathcal{S}_{k,k})$.

[4]There is a discrepancy between our presentation and the presentation given in Cremona [6], because Cremona works with the groups $\widetilde{\Gamma}_0(\mathfrak{N}) = \mathrm{SL}_2(\mathcal{O}) \cap \Gamma_0(\mathfrak{N})$ rather than $\Gamma_0(\mathfrak{N})$, which is more natural from an automorphic point of view. Note that $\Gamma_0(\mathfrak{N})/\widetilde{\Gamma}_0(\mathfrak{N})$ has order two, and is generated by J. Thus the characteristic zero cohomology of the orbifold $\mathcal{H}/\Gamma_0(\mathfrak{N})$ can be recovered from the cohomology of $\mathcal{H}/\widetilde{\Gamma}_0(\mathfrak{N})$ by taking J-invariants, and the space of forms we consider is precisely Cremona's + space.

The authors would like to thank William Stein and David Pollack, who computed the following data using the result above on December 15, 2004 and November 21, 2005 respectively; William Stein writing a program to compute the dimensions of these spaces, and David Pollack writing an additional routine to compute Hecke operators. The authors take full responsibility for the correctness of these results. As some independent confirmation of these computations, we note that part 1 is already known by previous computations of Cremona [6], and in part 3 the dimensions we compute agree with a known lower bound that is independent of the computation. Finally, for part two, we list the first few eigenvalues and note that they lie in $\mathbf{Q}$ (the space of cusp forms is one dimensional over $\mathbf{Q}$) and satisfy the (conjectural) Ramanujan bound $|a_{\mathfrak{p}}| \le 2N(\mathfrak{p})^{3/2}$.

Lemma 8.10. *Let $K = \mathbf{Q}(\sqrt{-2})$, let $\mathcal{O} = \mathcal{O}_K$, and let $\mathfrak{N} \subset \mathcal{O}$ be an ideal such that $N_{K/\mathbf{Q}}(\mathfrak{N})$ is squarefree and co-prime to 2. Then:*

1. *The space of cuspidal modular forms over K of weight $(2,2)$ and level $\Gamma_0(\mathfrak{N})$ is trivial for all such $\mathfrak{N}$ with $N_{K/\mathbf{Q}}(\mathfrak{N}) \le 337$ except for $\mathfrak{N}$ in the following list:*
$$\{7+\sqrt{-2}, 11+7\sqrt{-2}, 7+10\sqrt{-2}, 13+7\sqrt{-2}, 5+11\sqrt{-2}, 9+11\sqrt{-2}, 7+12\sqrt{-2}\}$$
of norms $51, 219, 249, 267, 323$ and 337 respectively.
2. *The space of cuspidal modular forms over K of weight $(4,4)$ and level $\Gamma_0(\mathfrak{N})$ is trivial for all such $\mathfrak{N}$ with $N_{K/\mathbf{Q}}(\mathfrak{N}) \le 337$ except for $\mathfrak{N} = 5+7\sqrt{-2}$ of norm 123. There is a unique cuspform g up to normalization of weight $(4,4)$ and level $\Gamma_0(5+7\sqrt{-2})$. The Hecke eigenvalues $a_{\mathfrak{p}}$ for $\mathfrak{p} \nmid \mathfrak{N}$ and $N(\mathfrak{p}) \le 25$ of g are given by the following table:*

$\mathfrak{p}$	$\sqrt{-2}$	$1-\sqrt{-2}$	$3+\sqrt{-2}$	$3-\sqrt{-2}$	$3+2\sqrt{-2}$	$3-2\sqrt{-2}$	5
$a_{\mathfrak{p}}$	0	4	-24	36	-54	-102	-118
$\lceil 2N(\mathfrak{p})^{3/2} \rceil$	6	11	73	73	141	141	250

3. *The space of cuspidal modular forms over K of weight $(2k,2k)$ and level $\Gamma_0(1)$ is equal to the space of forms of level one arising from base change from $\mathbf{Q}$ for all $2k \le 96$.*

Referring to the above table, note that the congruences $a_{\mathfrak{p}} \equiv 1 + N(\mathfrak{p})^3 \mod 4$ for $\mathfrak{p} \nmid 2$ and $a_{\mathfrak{p}} \equiv 1 + N(\mathfrak{p})^3 \mod 3$ appear to hold. (Since K is ramified at 2, being Eisenstein at 2 does not imply ordinariness). It might be interesting to determine the $\mathrm{GL}_2(\mathbf{F}_5)$-representation associated to g, although it could *a priori* be quite large.

A Generic Modules

A.1 Left and Right G-Modules

Let G be a finite group, and $D \subset G$ a subgroup. For $\mathcal{K}$ a field, denote the group ring G with coefficients in $\mathcal{K}$ by $\Lambda_{\mathcal{K}} := \mathcal{K}[G]$. We will view $\Lambda_{\mathcal{K}}$ as a $\mathcal{K}$-algebra, and therefore also as bi-module over itself. Let F denote a field of characteristic zero over which all irreducible representations of G over an algebraically closed field are defined, and let E be some extension of F. In our application, F will be a field algebraic over $\mathbf{Q}$ and E a localization of F at some prime above p. Let $\Lambda := \Lambda_E = E[G]$ so that we have the natural isomorphism of E-algebras $\Lambda \simeq \Lambda_F \otimes_F E$. A **left ideal** M of Λ is a left Λ-module equipped with an inclusion $M \hookrightarrow \Lambda$ of left Λ-modules, and a **right ideal** N of Λ is defined similarly.

Definition A.1. *For any subset M of Λ, let $\Lambda[M]$ denote the set of elements of Λ that are annihilated by M on the right; i.e.,*

$$\Lambda[M] := \{\lambda \in \Lambda \mid m\lambda = 0 \text{ for all } m \in M\}$$

and let $[M]\Lambda$ denote the set of elements of Λ that are annihilated by M on the left; i.e.,

$$[M]\Lambda := \{\lambda \in \Lambda \mid \lambda m = 0 \text{ for all } m \in M\}.$$

Lemma A.2.

- *If M is a right ideal of Λ then $\Lambda[M]$ is a left ideal.*
- *If M is a left ideal then $[M]\Lambda$ is a right ideal.*

For any left Λ-module W, the set $\mathrm{Hom}_G(\Lambda, W)$ inherits the structure of a left Λ-module where scalar multiplication $(\lambda, \phi) \mapsto \lambda \cdot \phi$ for $\lambda \in \Lambda$ and $\phi \in \mathrm{Hom}_G(\Lambda, W)$ is given by the rule $(\lambda \cdot \phi)(x) = \phi(x \cdot \lambda)$. The left Λ-module $\mathrm{Hom}_G(\Lambda, W)$ is then identified with W via the map i sending ϕ to $i(\phi) := \phi([1])$. In particular, taking $W = \Lambda$ the isomorphism i establishes the canonical identification $\phi \mapsto \phi([1])$ of the left Λ-module $\mathrm{Hom}_G(\Lambda, \Lambda)$ with Λ. Here, $\mathrm{Hom}_G(\Lambda, \Lambda)$ refers to the set of homomophisms $\phi : \Lambda \to \Lambda$ that preserve the left-Λ-module structure of Λ; explicitly $\phi(\lambda \cdot x) = \lambda \cdot \phi(x)$ for all $\lambda, x \in \Lambda$.

Lemma A.3. *Let M be a left ideal of Λ. The isomorphism i induces a commutative diagram*

$$\begin{tikzcd}
\mathrm{Hom}_G(\Lambda/M, \Lambda) \arrow[r] \arrow[d] & {[M]\Lambda} \arrow[d] \\
\mathrm{Hom}_G(\Lambda, \Lambda) \arrow[r] & \Lambda
\end{tikzcd}$$

where the vertical homomorphisms are the natural inclusions.

Proof. The homomorphism $\phi : \Lambda \to \Lambda$ lies in $\mathrm{Hom}_G(\Lambda/M, \Lambda)$ if and only if $\phi(m) = 0$ or equivalently if $(m \cdot \phi)([1]) = 0$, or $m \cdot i(\phi) = 0$, for all $m \in M$. $\square$

Lemma A.4. *The mapping*

$$M \quad \longmapsto \quad N := [M]\Lambda$$

from left ideals M to right ideals N, and the mapping

$$N \quad \longmapsto \quad M := \Lambda[N]$$

from right ideals N to left ideals M, are two-sided inverses of each other, and are one:one correspondences between the set of left ideals of Λ and the set of right ideals of Λ.

Proof. Let M be a left ideal and $N := [M]\Lambda$. It follows immediately from the definitions that $M \subseteq \Lambda[N]$, and thus to show that $M = \Lambda[N]$ we need only show $\dim(M) \geq \dim(\Lambda[N])$. From the representation theory of finite groups one sees that

$$\dim(M) + \dim(\mathrm{Hom}_G(\Lambda/M, \Lambda)) = \dim(\Lambda),$$

and using Lemma A.3 we have

$$\dim(M) + \dim([M]\Lambda) = \dim(\Lambda).$$

Similarly, using (only) that N is a right ideal, we have

$$\dim(N) + \dim(\Lambda[N]) = \dim(\Lambda).$$

Since $N = [M]\Lambda$ the lemma follows. $\square$

We have the natural E-linear functional $\iota : \Lambda \to E$ that associates to any element $\sum_{g \in G} a_g[g] \in \Lambda$ the coefficient of the identity element, $a_1 \in E$; i.e., $\iota(\sum_{g \in G} a_g[g]) = a_1$. For any left Λ-module W, composition with this functional ι induces a homomorphism

$$\mathrm{Hom}_\Lambda(W, \Lambda) \subset \mathrm{Hom}_E(W, \Lambda) \xrightarrow{\iota} \mathrm{Hom}_E(W, E) = W^\vee.$$

The flanking E-vector spaces, $\mathrm{Hom}_\Lambda(W, \Lambda) \to W^\vee$, both have natural right Λ-module structures defined as follows. On $\mathrm{Hom}_\Lambda(W, \Lambda)$ the right scalar multiplication $(\phi, \lambda) \mapsto \phi \cdot \lambda$ is given by the rule $(\phi \cdot \lambda)(x) = \phi(\lambda \cdot x)$, or equivalently, by letting $g \in G$ act on $\phi : W \to \Lambda$ by composition with $g^{-1} : \Lambda \to \Lambda$ and extending this action linearly to obtain a right Λ-module structure. We define the right Λ-structure on $W^\vee$ by the corresponding rule: for $\phi \in \mathrm{Hom}_E(W, E)$ and $g \in G$ define $\phi \cdot g$ by $(\phi \cdot g)(w) = \phi(g \cdot w)$ for $w \in W$.

Consider, as well, the homomorphism $W^\vee = \mathrm{Hom}_E(W, E) \xrightarrow{j} \mathrm{Hom}_E(W, \Lambda)$ that associates to $\phi \in \mathrm{Hom}_E(W, E)$ the homomorphism $j\phi \in \mathrm{Hom}_\Lambda(W, \Lambda)$ which, for $w \in W$ satisfies the formula

$$j\phi(w) = \sum_{g \in G} \phi(g \cdot w)[g]^{-1} \in \Lambda.$$

Lemma A.5. $\mathrm{Hom}_\Lambda(W, \Lambda) \xrightarrow{\iota} W^\vee$ *is an isomorphism of right* Λ*-modules and the homomorphism* j *described above is its inverse.*

Proof. This is a direct check. $\square$

In the special case where we take W to be Λ itself, viewed as Λ-bimodule via left and right multiplication, the above identifications offer us isomorphisms of bi-modules,

$$\mathrm{Maps}(G, E) \xrightarrow{\simeq} \mathrm{Hom}_E(\Lambda, E) \xrightarrow{\simeq} \Lambda \xrightarrow{\simeq} \mathrm{Hom}_G(\Lambda, \Lambda),$$

the composition of the first two of these isomorphisms being given by the rule $\phi \mapsto \sum_{g \in G} \phi(g) \cdot [g^{-1}]$ for $\phi \in \mathrm{Maps}(G, E)$, and the last by the rule $\lambda \mapsto \{x \mapsto \lambda \cdot x\}$. This latter isomorphism is indeed an isomorphism of *bi*-modules if we impose a left Λ-module structure on $\mathrm{Hom}_G(\Lambda, \Lambda)$ in the usual way—i.e., $(\lambda \cdot \phi)(x) = \lambda \cdot \phi(x)$—and if we give $\mathrm{Hom}_G(\Lambda, \Lambda)$ a right Λ-module structure by requiring $(\phi \cdot g)(x) = \phi(x \cdot g^{-1})$ for all $g \in G$ and extending linearly.

A consequence of Lemma A.5 is that for any left ideal $M \subset \Lambda$ we have a natural identification of right Λ-modules:

$$(\Lambda/M)^\vee \quad \stackrel{\iota}{\simeq} \quad \mathrm{Hom}_\Lambda(\Lambda/M,\Lambda) \quad = \quad [M]\Lambda.$$

If V is a left D-module for the subgroup $D \subset G$, then $V^\vee := \mathrm{Hom}_E(V,E)$ inherits the structure of a right D-module (via the usual formula $\phi \cdot g(v) = \phi(g \cdot v)$) and if V is a right D-module then $V^\vee$ is naturally a left D-module.

A.2 Generic Ideals

The module Λ has a canonical F-rational structure as a Λ-bi-module coming from the identification $\Lambda = \Lambda_F \otimes_F E$. Given a left or right ideal M of Λ, one can ask the extent to which M sits "perpendicularly" inside Λ with respect to this rational structure. If M is maximally skew, then we shall define M to be generic. More generally, we shall define such a notion relative to the subgroup $D \subset G$.

Definition A.6. *An arbitrary E-vector subspace $V \subseteq \Lambda$ is called a* right rational D-subspace *if there exists a right $F[D]$-submodule $V_F \subseteq \Lambda_F$ such that $V = V_F \otimes E$. One defines similarly the notion of* left rational D-subspace.

Lemma A.7. *If V is a right rational D-subspace, then $(\Lambda/V)^\vee = \mathrm{Hom}_E(\Lambda/V,E)$ is a left rational D-subspace.*

Proof. It suffices to note that if $(\Lambda/V)^\vee_F := \mathrm{Hom}_F(\Lambda_F/V_F,F)$ then $(\Lambda/V)^\vee_F$ is a left D-module with a natural inclusion into Λ_F, and $(\Lambda/V)^\vee = (\Lambda/V)^\vee_F \otimes E$. $\square$

Definition A.8. *A left ideal $M \subseteq \Lambda$ is* right generic *with respect to D if*

$$\dim(M \cap V) \le \dim(N \cap V)$$

for all right rational D-subspaces V and left ideals $N \subseteq \Lambda$ such that $M \simeq N$ as G-modules. If $M \subseteq \Lambda$ is a right ideal, then M is left generic with respect to D if the same formula holds for all left rational D-subspaces V and right ideals $N \simeq M$.

Example 1. *Let G be an abelian group, and D any subgroup. Then every irreducible representation of G occurs with multiplicity one inside Λ. In particular, if M and N are any two Λ-submodules with $M \simeq N$ then $M = N$, and so all such submodules are generic.*

Lemma A.9. *The left ideal M is right generic if and only if the right ideal $[M]\Lambda$ is left generic.*

Proof. For any $M \subseteq \Lambda$ there is an exact sequence and corresponding dimension formula:

$$0 \to M \cap V \to M \oplus V \to \Lambda \to \big((\Lambda/M)^\vee \cap (\Lambda/V)^\vee\big)^\vee \to 0,$$

$$\dim((\Lambda/M)^\vee \cap (\Lambda/V)^\vee) = \dim(M \cap V) - \dim(M) - \dim(V) + \dim(\Lambda).$$

From this formula (and its dual) it follows that M is generic if and only if $(\Lambda/M)^\vee$ is generic, since the dimensions of the intersections of $(\Lambda/M)^\vee$ with left rational D-modules can be explicitly compared to the intersection of M with right rational D-modules. Yet $(\Lambda/M)^\vee = [M]\Lambda$, so we are done. $\square$

A.3 First properties of generic modules

Genericity is not in general preserved under automorphisms of Λ. On the other hand, we have the following:

Lemma A.10. *Let $S : \Lambda \to \Lambda$ be an isomorphism of left Λ-modules sending $E[D]$ to itself. Then $S(M)$ is right generic with respect to D if and only if M is right generic with respect to D.*

Proof. Let V be a right rational D-subspace. Then $V = V.E[D]$. Thus the image of V under S is $V.E[D] = V$. Hence S preserves right rational D-subspaces, and the genericity of M follows from the following obvious formula:

$$\dim(S(M) \cap V) = \dim(S(M \cap V)) = \dim(M \cap V).$$

$\square$

We now extend our notion of generic to include certain submodules $M \subseteq P$, where P is a free Λ-module of rank one which does not necessarily come with a canonical generator. Let A be an "abstract" free left $E[D]$-module of rank one, without a canonical generator. Let $P = \Lambda \otimes_D A$. Clearly P is a free left Λ-module of rank one. Moreover, there is a natural class of isomorphisms from Λ to P given by maps of the form

$$T : \Lambda \to P, \qquad [1] \mapsto [1] \otimes a$$

for some generator a of A.

Definition A.11. *A left module $M \subseteq P$ is right generic with respect to D if for some generator $a \in A$ of the $E[D]$-module A, the left ideal $T^{-1}(M) \subseteq \Lambda$ is right generic with respect to D.*

Lemma A.12. *If $M \subseteq P$ is right generic with respect to D, then $T^{-1}(M) \subseteq \Lambda$ is right generic with respect to D for* all *a generating A.*

Proof. Given two such choices of generator a, a', it suffices to note that the composite: $S : T(a')^{-1} \circ T(a)$ preserves $E[D]$ and thus preserves generic ideals, by Lemma A.10. $\square$

A.4 Relative Homomorphism Groups

Let Y be a left Λ-module, and let $Z \subseteq Y$ be a vector subspace.

Definition A.13. *Let $\mathrm{Hom}_G(\Lambda, Y; Z)$ denote the G-equivariant homomorphisms ϕ from Λ to Y such that $\phi([1]) \in Z$. For a left ideal $M \subseteq \Lambda$, let $\mathrm{Hom}_G(\Lambda/M, Y; Z)$ denote the G-equivariant homomorphisms ϕ such that $\phi([1] \bmod M) \in Z$.*

Remark: The identification of $\mathrm{Hom}_G(\Lambda, Y)$ with Y identifies $\mathrm{Hom}_G(\Lambda, Y; Z)$ with Z.

Suppose that Y and Z have compatible rational structures, namely, there exists a left Λ_F-module Y_F, an F-vector subspace $Z_F \subseteq Y_F$ and isomorphisms $Y = Y_F \otimes_F E$, $Z = Z_F \otimes_F E$ compatible with the inclusion $Z \subseteq Y$. Then if M is a generic left ideal one may expect the homomorphism group $\mathrm{Hom}_G(\Lambda/M, Y; Z)$ to be 'as small as possible'. This expectation is borne out by the following result, which is the main form in which we apply our generic hypothesis.

Lemma A.14. *Let the left ideal $M \subseteq \Lambda$ be right generic with respect to D. Suppose that Y is a left Λ-module and $Z \subseteq Y$ a D-module subspace, and that Y and Z admit compatible rational structures. Suppose furthermore that Y admits a Λ-module injection $Y \to \Lambda$. Then for all left ideals $N \subseteq \Lambda$ with $M \simeq N$ as G-modules,*

$$\dim(\mathrm{Hom}_G(\Lambda/M, Y; Z)) \leq \dim(\mathrm{Hom}_G(\Lambda/N, Y; Z)).$$

Proof. Choose an injection $Y \hookrightarrow \Lambda$ compatible with rational structures. Such a map makes Z a rational left D-submodule of Λ. On the other hand one has the following identification:

$$\mathrm{Hom}_G(\Lambda/M, Y; Z)) = \mathrm{Hom}_G(\Lambda/M, Y) \cap \mathrm{Hom}_G(\Lambda, Y; Z) = \mathrm{Hom}_G(\Lambda/M, \Lambda) \cap \mathrm{Hom}_G(\Lambda, Y; Z).$$

By Lemma A.3 we may write this as $[M]\Lambda \cap Z$. By Lemma A.9, $[M]\Lambda$ is left generic with respect to D. Thus, as Z is a left rational D-module,

$$\dim(\mathrm{Hom}_G(\Lambda/M, Y; Z)) = \dim([M]\Lambda \cap Z) \leq \dim([N]\Lambda \cap Z) = \dim(\mathrm{Hom}_G(\Lambda/N, Y; Z)),$$

for any left $E[G]$-ideal N such that there exists a left $E[G]$-module isomorphism $N \simeq M$. $\square$

In order to apply this lemma, we shall describe exactly what vector subspaces of Y are of the form $\mathrm{Hom}_G(\Lambda/N, Y)$ for some left $E[G]$-ideal $N \subset \Lambda$. If $\mathrm{Irr}(G)$ denotes the set of irreducible representations of G, for each $i \in \mathrm{Irr}(G)$, fix a choice $V_{i,F}$ of a left $F[G]$-module that, as G-representation, corresponds to the irreducible representation i. Let $V_{i,F}^* := \mathrm{Hom}_F(V_{i,F}, F)$. Put $V_i := V_{i,F} \otimes_F E$ and $V_i^* := V_{i,F}^* \otimes_F E \simeq \mathrm{Hom}_E(V_i^*, E)$.

There is a canonical decomposition

$$\Lambda = \bigoplus_{i \in \mathrm{Irr}(G)} V_i \otimes V_i{}^*.$$

All (left) $E[G]$-modules Y can be written as $Y = \bigoplus_{i \in \mathrm{Irr}(G)} V_i \otimes T_i{}^*$, for some vector space $T_i{}^*$ with $\dim(T_i{}^*) = \dim(\mathrm{Hom}(Y, V_i))$. Requiring that Y admits an injective $E[G]$-module homomorphism into Λ is equivalent to insisting that $\dim(T_i{}^*) \leq \dim(V_i{}^*)$. Any left ideal N contained in $E[G]$ is then expressible as

$$N = \bigoplus_{i \in \mathrm{Irr}(G)} V_i \otimes T_i{}^* \subset \bigoplus_{i \in \mathrm{Irr}(G)} V_i \otimes V_i{}^*$$

where the T_i^* are E-vector subspaces $T_i^* \subset V_i{}^*$. Similarly, any right $E[G]$-module is of the form $\bigoplus_{i \in \mathrm{Irr}(G)} S_i \otimes V_i{}^*$.

If the left $E[G]$-module Y has a rational structure, an $E[G]$-module homomorphism $Y \to \Lambda$ is compatible with this structure if and only if the corresponding E-linear homomorphism $T_i{}^* \to V_i{}^*$ is compatible with the rational structures on these E-vector spaces.

Also, we may express the right $E[G]$-submodule $[N]\Lambda \subset \Lambda$ as

$$[N]\Lambda = \bigoplus S_i \otimes V_i{}^*.$$

Note that the isomorphism class of N as $E[G]$-module is completely determined by (and determines) the numbers $\dim(S_i)$ (for $i \in \mathrm{Irr}(G)$). Now consider the natural chain of inclusions of E-vector spaces

$$\mathrm{Hom}_G(\Lambda/N, Y) \subset \mathrm{Hom}_G(\Lambda, Y) = Y \subset \Lambda.$$

Viewing, then, both E-vector spaces $[N]\Lambda$ and $\mathrm{Hom}_G(\Lambda/N, Y)$ as subspaces of Λ, we have the formula

$$\mathrm{Hom}_G(\Lambda/N, Y) = Y \cap [N]\Lambda \subset \Lambda$$

giving us

$$\mathrm{Hom}_G(\Lambda/N, Y) = \left(\bigoplus V_i \otimes T_i^*\right) \cap \left(\bigoplus S_i \otimes V_i^*\right) = \bigoplus S_i \otimes T_i^* \subseteq \Lambda.$$

The content of Lemma A.14 is that if N is generic the subspaces $S_i \subseteq V_i$ are "maximally skew" to any rational D-submodule Z of Λ, in the sense that

$$\dim\left(Z \cap \bigoplus S_i \otimes T_i^*\right) \leq \dim\left(Z \cap \bigoplus S_i' \otimes T_i^*\right),$$

for any $S_i' \subseteq V_i$ with $\dim(S_i) = \dim(S_i')$.

If $Y' \subseteq Y$ is any E-vector subspace, define its Λ-annihilator in the evident way:

$$N(Y') := \{\lambda \in \Lambda \mid \lambda \cdot y' = 0 \text{ for all } y' \in Y'\}.$$

So, $N(Y') \subset \Lambda$ is a left ideal. Putting $N := N(Y')$ we have the inclusions

$$Y' \subset Y \cap [N]\Lambda = \mathrm{Hom}_G(\Lambda/N, Y)$$

and the following lemma provides a characterization of which E-vector subspaces Y' have the property that this inclusion is an isomorphism, i.e., it offers us a characterization of the E-vector subspaces Y' that are of the form $\mathrm{Hom}_\Lambda(\Lambda/N, Y) = Y \cap [N]\Lambda \subset Y$, for some left ideal $N \subset \Lambda$.

Let Y_i denote the V_i-isotypic component of Y. If $Y' \subseteq Y$ is an E-vector subspace and $i \in \mathrm{Irr}(G)$, we denote by Y_i' the intersection of Y' with the i-isotypic component of the left $E[G]$-module Λ, or equivalently, $Y_i' = Y' \cap Y_i$. Any irreducible sub-$E[G]$-module in $Y = \bigoplus_{j \in \mathrm{Irr}(G)} V_j \otimes {T_j}^*$ that is isomorphic to V_i is of the form $V_i \otimes \mathcal{L}_i$ for $\mathcal{L}_i \subset {T_j}^*$ for some one-dimensional E-subvector space of ${T_j}^*$.

Lemma A.15. *Let $Y' \subseteq Y$ be an E-vector subspace. The following bulleted statements are equivalent.*

- *For the left ideal $N = N(Y') \subset \Lambda$ the inclusion $Y' \subseteq \mathrm{Hom}_G(\Lambda/N, Y)$ described above is an isomorphism.*
- *There are E-vector subspaces $S_i \subset V_i$ such that the diagram*

$$\begin{tikzcd} Y' \arrow[r, equal] \arrow[d] & \oplus_i S_i \otimes T_i^* \arrow[d] \\ Y \arrow[r, equal] & \oplus_i V_i \otimes T_i^* \end{tikzcd}$$

is commutative, where the vertical morphisms are the natural inclusions.

- *1. Y' is generated by the subspaces $Y' \cap V$ as V ranges over the irreducible $E[G]$-submodules $V \subseteq Y$.*

  2. If $V, V' \subseteq Y$ are two irreducible $E[G]$-submodules of Y, and $\mathrm{Isom}(V, V') \neq \emptyset$*, then any isomorphism $V \to V'$ induces an isomorphism $Y' \cap V \to Y' \cap V'$.*

Furthermore, if Y' satisfies one, and hence all, of these bullets, then

$$\dim(S_i) = \dim(\mathrm{Hom}_G(\Lambda/N, V_i)) = \dim(V_i) - \dim(\mathrm{Hom}_G(N, V_i))$$

for any V_i with $\mathrm{Hom}_G(V_i, Y) \neq 0$.

Proof. First note that for any choice of E-vector subspaces $S_i \subset V_i$ (all i) there is a unique left ideal $N \subset \Lambda$ such that

$$[N]\Lambda = \bigoplus_i S_i \otimes V_i^* \subset \bigoplus_i V_i \otimes V_i^* = \Lambda$$

which (by the discussion just prior to the statement of our lemma) establishes the equivalence of the first two bullets. We now propose to show the equivalence of the last two bullets.

1. Suppose that $Y' = \bigoplus_i S_i \otimes T_i{}^* \subseteq Y$.

   By Schur's lemma, any left $E[G]$-automorphism of $Y_i = V_i \otimes T_i{}^*$ is induced from a bilinear isomorphism $V_i \times T_i{}^* \to V_i \times T_i{}^*$ that is *the identity* on the first factor.

   Then, as we have noted, any irreducible $E[G]$-submodule $V \subseteq Y$ is of the form $V_i \otimes \mathcal{L}_i$ for some one dimensional subspace $\mathcal{L}_i \subset T_i{}^*$. Since

   $$Y' \cap (V_i \otimes \mathcal{L}_i) = S_i \otimes \mathcal{L}_i,$$

   such spaces clearly generate Y'. Thus Y' satisfies the conditions of part (1).

   For part (2), we may write $V = V_i \otimes \mathcal{L}_i$ and $V' = V_i \otimes \mathcal{L}_i'$ for one dimensional subspaces $\mathcal{L}_i, \mathcal{L}_i' \subset T_i{}^*$. Any left $E[G]$-module isomorphism $V_i \otimes \mathcal{L}_i \to V_i \otimes \mathcal{L}_i'$ arises from a bilinear map $V_i \times T_i{}^* \to V_i \times T_i{}^*$ which is the identity on V_i and sends $\mathcal{L}_i$ to $\mathcal{L}_i'$. All such maps induce isomorphisms

   $$S_i \otimes \mathcal{L}_i = Y' \cap (V_i \otimes \mathcal{L}_i) \xrightarrow{\simeq} Y' \cap (V_i \otimes \mathcal{L}_i') = S_i \otimes \mathcal{L}_i'.$$

2. Suppose that Y' is a module satisfying conditions (1) and (2). Choose any one dimensional subspace $\mathcal{L}_i \subset T_i{}^*$, let $V = V_i \otimes \mathcal{L}_i \subset Y$, and let

   $$Y' \cap V = Y' \cap (V_i \otimes \mathcal{L}_i) =: S_i \otimes \mathcal{L}_i.$$

   If $\mathcal{L}_i' \subset T_i{}^*$ is any other one dimensional subspace, then there is a left $E[G]$-isomorphism $V_i \otimes T_i{}^* \to V_i \otimes T_i{}^*$ acting as the identity on the first factor and sending $V = V_i \otimes \mathcal{L}_i$ to $V' := V_i \otimes \mathcal{L}_i'$. Hence by condition (2) we deduce that $Y' \cap V' = Y' \cap (V_i \otimes \mathcal{L}_i') \simeq S_i \otimes \mathcal{L}_i'$, and letting $\mathcal{L}_i$ range over all one dimensional subspaces of T_i^* we conclude that $S_i \otimes T_i{}^* \subset Y_i'$, where Y_i' is the V_i-isotypic component of Y'. To deduce that this is an equality, recall by condition (1) that Y is generated by subspaces of the form $Y' \cap V$, and hence Y_i' is generated by subspaces of the form $Y' \cap V$ with $V \simeq V_i$. The above argument shows that all such intersections are all of the form $S_i \otimes \mathcal{L}_i$ for some $\mathcal{L}_i \subset T_i{}^*$, and hence $S_i \otimes T_i{}^* = Y_i'$, and we are done.

For the statements regarding dimensions of isotypic components, suppose that $\mathrm{Hom}_G(V_i, Y) \neq 0$. Then $T_i^* \neq 0$, and thus S_i is determined uniquely from $S_i \otimes T_i^*$. Since S_i is identified with the V_i-isotypic component of $[N]\Lambda$, the dimension formulas follow. $\square$

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